Open Problems in Algebraic General Topology*

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Abstract

This document lists in one place all conjectures and open problems in my Algebraic General Topology research which were yet not solved. This document also contains other relevant materials such as proved theorems related with the conjectures.

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Organizational info

Discuss these problems and their solutions in the `algebraic-general-topology` Google group.

See `http://www.mathematics21.org/algebraic-general-topology.html` for more details.

See also `http://filters.wikidot.com/open-problems` for open problems about filters.

Read http://www.mathematics21.org/solvers.html if you solved any of the below problems in order that I could nominate you for Abel Prize if I found your solutions worth it.

Misc

Conjecture 1. A reloid f is monovalued iff

$$\forall g \in \mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f) \colon (g \sqsubseteq f \Rightarrow \exists A \in \mathfrak{F}(\operatorname{Src} f) \colon g = f|_{\mathcal{A}}).$$

*. This document has been written using the GNU TEXMACS text editor (see `www.texmacs.org`).

Conjecture 2. The filtrator of funcoids is:

1. with separable core;

2. with co-separable core.

Conjecture 3. Let $\mho$ be a set, $\mathfrak{F}$ be the set of f.o. on $\mho$, $\mathfrak{P}$ be the set of principal f.o. on $\mho$, let n be an index set. Consider the filtrator $(\mathfrak{F}^n; \mathfrak{P}^n)$. Then if f is a completary multifuncoid of the form $\mathfrak{P}^n$, then $\uparrow\uparrow f$ is a completary multifuncoid of the form $\mathfrak{F}^n$.

Conjecture 4. Let f_1 and f_2 are monovalued, entirely defined funcoids with $\operatorname{Src} f_1 = \operatorname{Src} f_2 = A$. Then there exists a pointfree funcoid $f_1 \times^{(D)} f_2$ such that (for every fitler x on A)

$$\langle f_1 \times^{(D)} f_2 \rangle\, x = \bigsqcup \{ \langle f_1 \rangle\, X \times^{\mathsf{FCD}} \langle f_2 \rangle\, X \quad | \quad X \in \operatorname{atoms} x \}.$$

(The join operation is taken on the lattice of filters with reversed order.)

A positive solution of this problem may open a way to prove that some funcoids-related categories are cartesian closed.

Conjecture 5. $b \not\asymp^{\operatorname{Anch}(\mathfrak{A})} \operatorname{StarComp}(a; f) \Leftrightarrow \forall A \in \operatorname{GR} a, B \in \operatorname{GR} b, i \in n\colon A_i\, [f_i]\, B_i$ for anchored relations a and b on powersets.

It's consequence:

Conjecture 6. $b \not\asymp^{\operatorname{Anch}(\mathfrak{A})} \operatorname{StarComp}(a; f) \Leftrightarrow a \not\asymp^{\operatorname{Anch}(\mathfrak{A})} \operatorname{StarComp}(b; f^\dagger)$ for anchored relations a and b on powersets.

Conjecture 7. $b \not\asymp^{\mathsf{Strd}(\mathfrak{A})} \operatorname{StarComp}(a; f) \Leftrightarrow a \not\asymp^{\mathsf{Strd}(\mathfrak{A})} \operatorname{StarComp}(b; f^\dagger)$ for pre-staroids a and b on powersets.

Conjecture 8. $f \sqsubseteq \prod^{\mathsf{RLD}} a \Leftrightarrow \forall i \in \operatorname{arity} f\colon \operatorname{Pr}_i^{\mathsf{RLD}} f \sqsubseteq a_i$ for every multireloid f and $a_i \in \mathfrak{F}((\operatorname{form} f)_i)$ for every $i \in \operatorname{arity} f$.

Conjecture 9. $L \in [f] \Rightarrow [f] \cap \prod_{i \in \operatorname{dom} \mathfrak{A}} \operatorname{atoms} L_i \neq \emptyset$ for every pre-multifuncoid f of the form whose elements are atomic posets. (Does this conjecture hold for the special case of form whose elements are posets on filters on a set?)

Conjecture 10. The formula $f \sqcup^{\mathsf{FCD}(\mathfrak{A})} g \in \mathsf{cFCD}(\mathfrak{A})$ is not true in general for completary multifuncoids (even for multifuncoids on powersets) f and g of the same form $\mathfrak{A}$.

Conjecture 11. $\operatorname{GR} \operatorname{StarComp}(a \sqcup^{\mathsf{pFCD}} b; f) = \operatorname{GR} \operatorname{StarComp}(a; f) \sqcup^{\mathsf{pFCD}} \operatorname{GR} \operatorname{StarComp}(b; f)$ if f is a pointfree funcoid and a, b are multifuncoids of the same form, composable with f.

Conjecture 12. Every metamonovalued funcoid is monovalued.

Conjecture 13. Every metamonovalued reloid is monovalued.

Conjecture 14. Every monovalued reloid is metamonovalued.

Problem 15. Let A and B be infinite sets. Characterize the set of all coatoms of the lattice $\mathsf{FCD}(A;\, B)$ of funcoids from A to B. Particularly, is this set empty? Is $\mathsf{FCD}(A;\, B)$ a coatomic lattice? coatomistic lattice?

Hyperfuncoids

Let $\mathfrak{A}$ be an indexed family of sets.

Products are $\prod A$ for $A \in \prod \mathfrak{A}$.

Hyperfuncoids are filters $\mathfrak{F}\Gamma$ on the lattice Γ of all finite unions of products.

Problem 16. Is $\bigsqcap^{\mathsf{FCD}}$ a bijection from hyperfuncoids $\mathfrak{F}\Gamma$ to:

1. prestaroids on $\mathfrak{A}$;
2. staroids on $\mathfrak{A}$;
3. completary staroids on $\mathfrak{A}$?

If yes, is up^{Γ} defining the inverse bijection?

If not, characterize the image of the function $\bigsqcap^{\mathsf{FCD}}$ defined on $\mathfrak{F}\Gamma$.

Provability without axiom of choice

Conjecture 17. Distributivity of the lattice $\mathsf{FCD}(A; B)$ of funcoids (for arbitrary sets A and B) is not provable in ZF (without axiom of choice).

Conjecture 18. $a \setminus^* b = a \# b$ for arbitrary filters a, b on powersets is not provable in ZF (without axiom of choice).

Complete funcoids and reloids

Question 19. Is $\mathrm{Compl}\mathsf{FCD}(A; B)$ a co-brouwerian lattice?

Conjecture 20. Composition of complete reloids is complete.

Conjecture 21. $\mathrm{Compl}\, f \sqcap \mathrm{Compl}\, g = \mathrm{Compl}(f \sqcap g)$ for every reloids $f, g \in \mathsf{RLD}(A; B)$ (for every sets A, B).

Conjecture 22. $\mathrm{Compl}\, f = f \setminus^* (\Omega(\mathrm{Src}\, f) \times^{\mathsf{FCD}} 1^{\mathfrak{F}(\mathrm{Dst}\, f)})$ for every funcoid f.

Conjecture 23. $\mathrm{Compl}\, f = f \setminus^* (\Omega(\mathrm{Src}\, f) \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\mathrm{Dst}\, f)})$ for every reloid f.

Question 24. Is $\mathrm{Compl}\mathsf{RLD}(A;\, B)$ a distributive lattice? Is $\mathrm{Compl}\mathsf{RLD}(A;\, B)$ a co-brouwerian lattice? (for every sets A and B).

Conjecture 25. Let A, B, C be sets. If $f \in \mathsf{RLD}(B; C)$ is a complete reloid and $R \in \mathscr{P}\mathsf{RLD}(A; B)$ then $f \circ \bigsqcup R = \bigsqcup \langle f \circ \rangle R$.

Conjecture 26. Every entirely defined monovalued isomorphism of the category of funcoids is a discrete funcoid.

Conjecture 27. For composable reloids f and g it holds

1. $\operatorname{Compl}(g \circ f) = (\operatorname{Compl} g) \circ f$ if f is a co-complete reloid;
2. $\operatorname{CoCompl}(f \circ g) = f \circ \operatorname{CoCompl} g$ if f is a complete reloid;
3. $\operatorname{CoCompl}((\operatorname{Compl} g) \circ f) = \operatorname{Compl}(g \circ (\operatorname{CoCompl} f)) = (\operatorname{Compl} g) \circ (\operatorname{CoCompl} f)$;
4. $\operatorname{Compl}(g \circ (\operatorname{Compl} f)) = \operatorname{Compl}(g \circ f)$;
5. $\operatorname{CoCompl}((\operatorname{CoCompl} g) \circ f) = \operatorname{CoCompl}(g \circ f)$.

Relationships of funcoids and reloids

Conjecture 28. $(\mathsf{RLD})_\Gamma f = (\mathsf{RLD})_{\mathrm{in}} f$ for every funcoid f.

Conjecture 29. $(\mathsf{RLD})_{\mathrm{in}}(g \circ f) = (\mathsf{RLD})_{\mathrm{in}} g \circ (\mathsf{RLD})_{\mathrm{in}} f$ for every composable funcoids f and g.

Conjecture 30. $(\mathsf{RLD})_{\mathrm{out}} \mathrm{id}_{\mathcal{A}}^{\mathsf{FCD}} = \mathrm{id}_{\mathcal{A}}^{\mathsf{RLD}}$ for every filter $\mathcal{A}$.

Conjecture 31. $(\mathsf{RLD})_{\mathrm{in}}$ is not a lower adjoint (in general).

Conjecture 32. $(\mathsf{RLD})_{\mathrm{out}}$ is neither a lower adjoint nor an upper adjoint (in general).

Conjecture 33. If $\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \sqsubseteq (\mathsf{RLD})_{\mathrm{in}} f$ then $\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \sqsubseteq f$ for every funcoid f and $\mathcal{A} \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{B} \in \mathfrak{F}(\operatorname{Dst} f)$.

Conjecture 34. $\rho \bigsqcap F = \bigsqcap \langle \rho \rangle F$ for a set F of reloids. (ρ is defined in [1])

Conjecture 35. For every funcoid g

1. $\operatorname{Cor} (\mathsf{RLD})_{\mathrm{in}} g = (\mathsf{RLD})_{\mathrm{in}} \operatorname{Cor} g$;
2. $\operatorname{Cor} (\mathsf{RLD})_{\mathrm{out}} g = (\mathsf{RLD})_{\mathrm{out}} \operatorname{Cor} g$.

Conjecture 36. For every composable funcoids f and g

$$(\mathsf{RLD})_{\mathrm{out}}(g \circ f) \sqsupseteq (\mathsf{RLD})_{\mathrm{out}}\, g \circ (\mathsf{RLD})_{\mathrm{out}}\, f.$$

Connectedness of funcoids and reloids

Conjecture 37. A filter $\mathcal{A}$ is connected regarding a funcoid μ iff $\mathcal{A}$ is connected for every discrete funcoid $F \in \operatorname{up} \mu$.

Conjecture 38. A filter $\mathcal{A}$ is connected regarding a reloid f iff it is connected regarding the funcoid $(\mathsf{FCD}) f$.

Conjecture 39. Let $\mathcal{A}$ is a filter and F is a binary relation on $A \times B$ for some sets A, B. $\mathcal{A}$ is connected regarding $\uparrow^{\mathsf{FCD}(A;B)} F$ iff $\mathcal{A}$ is connected regarding $\uparrow^{\mathsf{RLD}(A;B)} F$.

Proposition 40. The following statements are equivalent for every endofuncoid μ and a set U:

1. U is connected regarding μ.

2. For every $a, b \in U$ there exists a totally ordered set $P \subseteq U$ such that $\min P = a$, $\max P = b$, and for every partion $\{X, Y\}$ of P into two sets X, Y such that $\forall x \in X, y \in Y : x < y$, we have $X\,[\mu]^*\,Y$.

Algebraic properties of S and S^*

Conjecture 41. $S(S(f)) = S(f)$ for

1. any endo-reloid f;

2. any endo-funcoid f.

Conjecture 42. For any endo-reloid f

1. $S(f) \circ S(f) = S(f)$;

2. $S^*(f) \circ S^*(f) = S^*(f)$;

3. $S(f) \circ S^*(f) = S^*(f) \circ S(f) = S^*(f)$.

Conjecture 43. $S(f) \circ S(f) = S(f)$ for any endo-funcoid f.

Oblique products

Conjecture 44. $\mathcal{A} \times_F^{\mathsf{RLD}} \mathcal{B} \sqsubset \mathcal{A} \ltimes \mathcal{B}$ for some f.o. $\mathcal{A}$, $\mathcal{B}$.

A stronger conjecture:

Conjecture 45. $\mathcal{A} \times_F^{\mathsf{RLD}} \mathcal{B} \sqsubset \mathcal{A} \ltimes \mathcal{B} \sqsubset \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for some f.o. $\mathcal{A}$, $\mathcal{B}$. Particularly, is this formula true for $\mathcal{A} = \mathcal{B} = \Delta \cap \uparrow^{\mathbb{R}}(0; +\infty)$?

Products

Conjecture 46. Cross-composition product (for small indexed families of reloids) is a quasi-cartesian function (with injective aggregation) from the quasi-cartesian situation $\mathfrak{S}_0$ of reloids to the quasi-cartesian situation $\mathfrak{S}_1$ of pointfree funcoids over posets with least elements.

Remark 47. The above conjecture is unsolved even for product of two multipliers.

Conjecture 48. $a \left[\prod^{(C)} f \right] b \Leftrightarrow \forall i \in \operatorname{dom} f : \operatorname{Pr}_i^{\mathsf{FCD}}\, a\, [f_i]\, \operatorname{Pr}_i^{\mathsf{FCD}}\, b$ for every indexed family f of funcoids and $a \in \mathsf{FCD}(\lambda i \in \operatorname{dom} f : \operatorname{Src} f_i)$, $b \in \mathsf{FCD}(\lambda i \in \operatorname{dom} f : \operatorname{Dst} f_i)$.

Conjecture 49. $\uparrow^{\mathsf{FCD}} A \left[\prod^{(C)} f \right] \uparrow^{\mathsf{FCD}} B \Leftrightarrow \uparrow^{\mathsf{RLD}} A \left[\prod^{(A)} f \right] \uparrow^{\mathsf{RLD}} B$ for every indexed family f of funcoids and $a \in \mathscr{P} \prod_{i \in \operatorname{dom} f} \operatorname{Src} f_i$, $a \in \mathscr{P} \prod_{i \in \operatorname{dom} f} \operatorname{Dst} f_i$.

Conjecture 50. $\left\langle \prod^{(A)} f \right\rangle \uparrow^{\mathsf{RLD}} X = (\mathsf{RLD})_{\mathrm{in}} \left\langle \prod^{(C)} f \right\rangle \uparrow^{\mathsf{FCD}} X$ for every indexed family f of funcoids and a suitable set X.

Compactness and Heine-Cantor theorem

Theorem 51. Let f be a T_1-separable compact reflexive symmetric funcoid and g be a reloid such that

1. $(\mathsf{FCD}) g = f$;
2. $g \circ g^{-1} \sqsubseteq g$.

Then $g = \langle f \times f \rangle^* \Delta$.

About the above conjecture see also

http://www.openproblemgarden.org/op/direct_proof_of_a_theorem_about_compact_funcoids

$\forall \mathcal{F} \in \mathfrak{F} \colon (\mathcal{F} \cap \operatorname{im} f \neq \emptyset \Rightarrow \exists \alpha \colon \{\alpha\} [f] \mathcal{F})$ or equivalently

$$\forall \mathcal{F} \in \mathfrak{F} \colon (\langle f^{-1} \rangle \mathcal{F} \neq \emptyset \Rightarrow \exists \alpha \colon \{\alpha\} \subseteq \langle f^{-1} \rangle \mathcal{F})$$

is a possible definition of *compact* funcoid. (A special case of this definition was hinted by Victor Petrov.) How this is related with open covers and finite covers from the traditional definition of compactness? Does compactness imply completeness?

Generalize Heine-Cantor theorem for funcoids and reloids.

Category theory related

Conjecture 52. The categories Fcd and Rld are cartesian closed (actually two conjectures).

Bibliography

[1] Victor Porton. Distributivity of compositon with a principal reloid over join of reloids. At http://www.mathematics21.org/binaries/decomposition.pdf.