This file is with partial proofs (with rough gibberish) about open problems I have tried to solve but have failed. If you solve something of this please notify me by email `porton@narod.ru`.

Conjecture 1. *$(\mathsf{RLD})_{\mathrm{out}}(g\circ f)\sqsupseteq(\mathsf{RLD})_{\mathrm{out}}\,g\circ(\mathsf{RLD})_{\mathrm{out}}\,f$ ($\sqsubseteq$ or $\sqsupseteq$) for composable funcoids f and g.*

Proof. $(\mathsf{RLD})_{\mathrm{out}}\,g\circ(\mathsf{RLD})_{\mathrm{out}}\,f=\bigsqcap^{\mathsf{RLD}}_{G\in\mathrm{up}\,g}G\circ\bigsqcap^{\mathsf{RLD}}_{F\in\mathrm{up}\,f}F\sqsubseteq G\circ F$ for every $F\in\mathrm{up}\,f$ and $G\in\mathrm{up}\,g$.

Thus $(\mathsf{RLD})_{\mathrm{out}}\,g\circ(\mathsf{RLD})_{\mathrm{out}}\,f\sqsubseteq\bigsqcap^{\mathsf{RLD}}_{F\in\mathrm{up}\,f,G\in\mathrm{up}\,g}(G\circ F)\sqsubseteq$?? [FIXME: Opposite inequalities!]

$(\mathsf{RLD})_{\mathrm{out}}\ (g\circ f)=\bigsqcap^{\mathsf{RLD}}\mathrm{up}(g\circ f)=\,??=\bigsqcap^{\mathsf{RLD}}_{F\in\mathrm{up}\,f,G\in\mathrm{up}\,g}(G\circ F)\sqsupseteq\bigsqcap^{\mathsf{RLD}}_{G\in\mathrm{up}\,(\mathsf{RLD})_{\mathrm{out}}g,F\in\mathrm{up}\,(\mathsf{RLD})_{\mathrm{out}}\,f}(G\circ F)=\bigsqcap^{\mathsf{RLD}}_{G\in\mathrm{up}\bigsqcap^{\mathsf{RLD}}\mathrm{up}\,g,F\in\mathrm{up}\bigsqcap^{\mathsf{RLD}}\mathrm{up}\,f}(G\circ F)=\bigsqcap^{\mathsf{RLD}}\mathrm{up}\,g\circ\bigsqcap^{\mathsf{RLD}}\mathrm{up}\,f$ □

Proposition 2. *A product of nonempty posets is a dcpo if and only iff each multiplier is a dcpo.*

Proof. [TODO: More detailed proof] Suppose one multiplier is not a dcpo. Take a chain with fixed elements (thanks our posets are nonempty) from other multipliers and for this multiplier take the values which form a chain without the join. This proves that the product is not a dcpo.

Now take that all multipliers are dcpo. Take a chain that is a set for which holds $a\sqsubseteq b\wedge b\sqsubseteq a\Rightarrow a=b$.

We have

??

Take an element t of the chain.

For each k we have $a'_i=\begin{cases}a_i & \text{if } i=k\\ t_i & \text{if } i\neq k\end{cases}$ and $b'_i=\begin{cases}b_i & \text{if } i=k\\ t_i & \text{if } i\neq k\end{cases}$

a' ?? belongs to the chain?

??We have $a_i\sqsubseteq b_i\wedge b_i\sqsubseteq a_i\Rightarrow a_i=b_i$ for all a, b in the chain. Take

Let . Take

??

It is a chain componentwise because the predicate $a\sqsubseteq b\wedge b\sqsubseteq a\Rightarrow a=b$ holds?? componentwise. Thus every component has a (calculated componentwise) join. □

Conjecture 3. *Funcoids f from A to B bijectively corresponds to the sets R of pairs $(\mathcal{X};\mathcal{Y})$ of filters (on A and B correspondingly) that*

1. *R is nonempty.*
2. *R is a lower set.*
3. *R is a dcpo (or what is the same product of two dcpos)*
4. *We can add axiom: $\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\sqsubseteq\bigsqcup\{\mathcal{X}\times^{\mathsf{FCD}}\mathcal{Y}\mid(\mathcal{X};\mathcal{Y})\in R_0\}\Rightarrow(\mathcal{A};\mathcal{B})\in R$*

by the mutually inverse formulas:

$$(\mathcal{X};\mathcal{Y})\in R\Leftrightarrow\mathcal{X}\times^{\mathsf{FCD}}\mathcal{Y}\sqsubseteq f \tag{1}$$

$$f=\bigsqcup\{\mathcal{X}\times^{\mathsf{FCD}}\mathcal{Y}\mid(\mathcal{X};\mathcal{Y})\in R\}. \tag{2}$$

Proof. Let R be defined by formula (1). That R is a nonempty lower set is obvious. Let's prove that R is a dcpo.

Let T be a chain in R and $\forall(\mathcal{A};\mathcal{B})\in T:\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\sqsubseteq f$.

$\forall(\mathcal{A};\mathcal{B})\in T:(\mathcal{X}\not\asymp\mathcal{A}\Rightarrow\mathcal{B}\sqsubseteq\langle f\rangle\mathcal{X})$

taking join we have:

$\mathcal{X}\not\asymp\mathcal{A}\Rightarrow\bigsqcup_{B\in\mathrm{Pr}_1T}B\sqsubseteq\langle f\rangle\mathcal{X}$

$\mathcal{A}\times^{\mathsf{FCD}}\bigsqcup_{B\in\mathrm{Pr}_1T}B\sqsubseteq f$

Repeating this, we get $\bigsqcup_{A\in\mathrm{Pr}_0T}A\times^{\mathsf{FCD}}\bigsqcup_{B\in\mathrm{Pr}_1T}B\sqsubseteq f$. Thus R is a dcpo.

It remains to prove that the formulas are mutually inverse.

Let f_0 be a funcoid, R be induced by f_0 by formula (1), f_1 be induced by R by formula (2). We will prove that $f_1=f_0$.

$f_1 = \bigsqcup \{\mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y} \mid \mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y} \sqsubseteq f_0\}$ what is equal to f_0 because the lattice of funcoids is atomistic and every atom is a funcoidal product.

Let now R_0 be a set of pairs of filters conformung to our axioms, f be a funcoid induced by R by formula (2), R_1 be a set of pairs of filters induced by f by formula (1). We will prove $R_1 = R_0$.

$(\mathcal{X}; \mathcal{Y}) \in R_1 \Leftrightarrow \mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y} \sqsubseteq \bigsqcup \{\mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y} \mid (\mathcal{X}; \mathcal{Y}) \in R_0\} \Leftarrow (\mathcal{X}; \mathcal{Y}) \in R_0$

$\mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y} \sqsubseteq \bigsqcup \{\mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y} \mid (\mathcal{X}; \mathcal{Y}) \in R_0\} \Rightarrow (\mathcal{X}; \mathcal{Y}) \in R_0$ because ??. [FIXME: It seems we need additional axioms!] □

Theorem 4. *If a funcoid is weakly metamonovalued, then it is monovalued.*

Proof. We have $(g \sqcap h) \circ f = (g \circ f) \sqcap (h \circ f)$ for every reloids g, h. We need to prove that f is monovalued.

Prove that exists $F \in \operatorname{up} f$ such that for every g, h we have $(g \sqcap h) \circ F = (g \circ F) \sqcap (h \circ F)$.

it's enough to prove that $(g \sqcap h) \circ F \sqsupseteq (g \circ F) \sqcap (h \circ F)$

Really, ??

thus F is monovalued.

$f \circ f^{-1} = \bigsqcap \{F \circ F^{-1} \mid F \in \operatorname{up} f\}$ □

Proposition 5. *The following statements are equivalent for every endofuncoid μ and a set U:*

1. *U is connected regarding μ.*
2. *For every $a, b \in U$ there exists a totally ordered set $P \subseteq U$ such that $\min P = a$, $\max P = b$, and for every partion $\{X, Y\}$ of P into two sets X, Y such that $\forall x \in X, y \in Y: x < y$, we have $X\, [\mu]^*\, Y$.*

Proof.

⇐. Let $A, B \in \mathscr{P} U$ are nonempty. We need to prove $A\, [\mu]^*\, B$. We can assume without loss of generality that $A \cap B = \emptyset$. Because A and B are nonempty, we can take $a \in A$ and $b \in B$. ??

⇒. The case $a = b$ is trivial. Assume $a \neq b$. Take $A, B \in \mathscr{P} U$ such that $a \in A$, $b \in B$ and $A \cup B = U$ and $A \cap B = \emptyset$. Then take orders P_A on A with $\min P_A = a$ and P_B on B with $\max P_B = b$. Then the poset $P = P_A + P_B$?? □

1 Directed funcoids

Let $[-\infty; +\infty]$ be the extended real line with the complete funcoid induced by the usual topology on this set.

Proposition 6. *Every ultrafilter on $[-\infty; +\infty]$ converges to exactly one point.*

Proof. It is a well known fact. □

Below is wrong: http://math.stackexchange.com/q/1874451/4876

[FIXME: below is wrong] Use http://math.stackexchange.com/a/1874862/4876

[TODO: $\Delta_+ \rightarrow \Delta_>$ or $\Delta_\geq$]

[TODO: Compare without explicit formulas.]

Lemma 7. *For every ultrafilter a and its limit point x:*

0. *$\langle [-\infty; +\infty] \rangle a = \Delta(x)$ if $x = a$*
1. *$\langle [-\infty; +\infty] \rangle a = \Delta_+(x)$ if $x > a$*
2. *$\langle [-\infty; +\infty] \rangle a = \Delta_-(x)$ if $x < a$*

Proof. 0. Obvious.

1. $\langle [-\infty; +\infty] \rangle a = \bigsqcap_{A \in \operatorname{up} a} \langle [-\infty; +\infty] \rangle^* A = \bigsqcap_{A \in \operatorname{up} a, A \sqsubseteq]x; +\infty]} \langle [-\infty; +\infty] \rangle^* A = \bigsqcap_{A \in \operatorname{up} a, A \sqsubseteq]x; +\infty]} \bigsqcup_{y \in A} \Delta(y)$

Because every $y > x$, we have $\Delta(y) \sqsubseteq]x; +\infty]$ and thus $\langle[-\infty; +\infty]\rangle a \sqsubseteq]x; +\infty]$.

It is clear that $\bigsqcup_{y\in A} \Delta(y) \sqsupseteq \Delta_+(x)$ and $\langle[-\infty; +\infty]\rangle a \sqsubseteq \Delta_+(x)$. Thus $\langle[-\infty; +\infty]\rangle a = \Delta_+(x)$.

2. $\langle[-\infty; +\infty]\rangle a = \bigsqcap_{A\in \operatorname{up} a} \langle[-\infty; +\infty]\rangle^* A = \bigsqcap_{A\in \operatorname{up} a, A\sqsubseteq[-\infty;x[} \langle[-\infty; +\infty]\rangle^* A = \bigsqcap_{A\in \operatorname{up} a, A\sqsubseteq[-\infty;x[} \bigsqcup_{y\in A} \Delta(y)$

Because every $y < x$ we have $\Delta(y) \sqsubseteq [-\infty; x[$ and thus $\langle[-\infty; +\infty]\rangle a \sqsubseteq [-\infty; x[$.

It is clear that $\bigsqcup_{y\in A} \Delta(y) \sqsupseteq \Delta_-(x)$ and $\langle[-\infty; +\infty]\rangle a \sqsubseteq \Delta_-(x)$. Thus $\langle[-\infty; +\infty]\rangle a = \Delta_-(x)$.

[TODO: More detailed proof.] □

Corollary 8. *For every ultrafilter a and its limit point x:*

$\langle[-\infty; +\infty] \sqcap \geq\rangle a = \Delta_+(x)$ *if* $x \geq a$

$\langle[-\infty; +\infty] \sqcap \geq\rangle a = \bot$ *if* $x < a$

Proof. Take into account $\langle[-\infty; +\infty] \sqcap \geq\rangle a = \langle[-\infty; +\infty]\rangle a \sqcap \langle\geq\rangle a$. □

Lemma 9. . *Then for every ultrafilter a and its limit point x:*

0. ??
1. $\left\langle\overrightarrow{[-\infty; +\infty]}\right\rangle a = \Delta_+(x)$ *if* $x \geq a$
2. $\left\langle\overrightarrow{[-\infty; +\infty]}\right\rangle a = \Delta_-(x)$ *if* $x < a$

Proof. $\langle R\rangle a = \bigsqcap_{A\in \operatorname{up} a} \langle R\rangle^* A = \bigsqcap_{A\in \operatorname{up} a} \bigsqcup_{y\in A} \Delta_+(y)$

1. ??

2. $\langle R\rangle a = \bigsqcap_{A\in \operatorname{up} a, A\sqsubseteq[-\infty;x[} \bigsqcup_{y\in A} \Delta_+(y) = ??$

$\bigsqcup_{y\in A} \Delta_+(y) \sqsupseteq \Delta_-(x)$; $\bigsqcup_{y\in A, A\sqsubseteq[-\infty;x[} \Delta_+(y) \sqsupseteq \Delta_-(x)$

$\langle R\rangle a \sqsubseteq \Delta_-(x)$ is obvious. □

Theorem 10. $[-\infty; +\infty] \sqcap^{\mathsf{FCD}} > \neq \overrightarrow{[-\infty; +\infty]}$.

Proof. $\langle[-\infty; +\infty] \sqcap \geq\rangle a = \bot$ if $x < a$ but $\left\langle\overrightarrow{[-\infty; +\infty]}\right\rangle a = \Delta_-(x)$ if $x < a$. □

Proposition 11. $\vec{f} = \operatorname{Compl}(f \sqcap >)$.

Proof. ?? □

2 Entourages of product of funcoids

Lemma 12. $\forall H \in \operatorname{up}(g \circ f) \exists F \in \operatorname{up} f : H \sqsupseteq g \circ F$

Proof. $H \sqsupseteq g \circ F \Leftrightarrow H \circ f^{-1\circ} \sqsupseteq g$ (proposition 1813). But $f^{-1\circ} \sqsupseteq \overline{f^{-1}}$, thus $H \sqsupseteq g \circ F \Leftarrow H \circ \overline{f^{-1}} \sqsupseteq g$. Take $G = H \circ \overline{f^{-1}}$, but $H \circ \overline{f^{-1}} \notin \operatorname{up} g$ because $g \circ f \circ \overline{f^{-1}} \sqsupseteq g$ does not hold by http://math.stackexchange.com/a/1862976/4876

??

[FIXME: This was proof of another lemma.]

Instead of funcoids we will assume that f and g are filters on Γ (they are isomorphic).

$g \circ f = \bigsqcap^{\mathfrak{F}\Gamma} \operatorname{up}^\Gamma(g \circ f) = \bigsqcap^{\mathfrak{F}\Gamma} ((\operatorname{up}^\Gamma g) \circ (\operatorname{up}^\Gamma f)) = \bigsqcap^{\mathfrak{F}\Gamma}_{F\in \operatorname{up}^\Gamma f, G\in \operatorname{up}^\Gamma g} (G \circ F)$ (lemma 1274).

$g \circ f = \bigsqcap^{\mathfrak{F}\Gamma}_{F\in \operatorname{up} f, G\in \operatorname{up} g} (G \circ F)$ (follows from above or directly from theorm 781)

$\{G \circ F \mid F \in \operatorname{up} f, G \in \operatorname{up} g\}$ is?? a generalized filter base on Γ.

Thus by properties of generalized filter bases (on Γ) for every $H \in \operatorname{up}^\Gamma(g \circ f)$ there exists $F \in \operatorname{up}^\Gamma f$, $G \in \operatorname{up}^\Gamma g$ such that $G \circ F \sqsubseteq H$.

??

Attempt to construct it: [Example: $f = 1$, choose $F_{H'} = 1$ and $G_{H'} = H'$. Then $\bigsqcap_{H'\in \operatorname{up}^\Gamma H} G_{H'} = \bigsqcap_{H'\in \operatorname{up}^\Gamma H} H' = \operatorname{Cor} H \in \operatorname{up} f$.]

Represent H as a meet of elements of Γ ($H = \bigsqcap \operatorname{up}^{\Gamma} H$). For each $H' \in \operatorname{up}^{\Gamma} H$ choose $F_{H'} \in \operatorname{up}^{\Gamma} f$, $G_{H'} \in \operatorname{up}^{\Gamma} g$ such that $H' \sqsupseteq G_{H'} \circ F_{H'}$. Moreover we can choose maximal $F_{H'}$, $G_{H'}$ such that this inequally holds. Then take $F = \bigsqcap_{H' \in \operatorname{up}^{\Gamma} H} F_{H'}$ and $G = \bigsqcap_{H' \in \operatorname{up}^{\Gamma} H} G_{H'}$. [FIXME: Does $F \in \operatorname{up} f$?] [TODO: If this does not work, then seems that there is a counter-example, because it is the stongest.]

[TODO: We MUST take maximal rather than arbitrary $F_{H'}, G_{H'}$. Otherwise take $f = 1$, $g = \mathrm{id}_{\Omega}$. Then if we take $F_{H'} = 1$ and replace all possible $G_{H'} \rightarrow G_{H'} \setminus \{(a; b)\}$, then $G = \bot \notin \operatorname{up} g$]

??

$H \in \operatorname{up}(g \circ f) \Rightarrow \langle H \rangle^{\ast} X \sqsupseteq \langle g \circ f \rangle^{\ast} X = \langle g \rangle \langle f \rangle^{\ast} X$.

For every X take $Y_X \in \operatorname{up} \langle f \rangle^{\ast} X$ such that $\langle g \rangle Y_X \sqsubseteq \langle H \rangle^{\ast} X$. If g is complete, we may assume that Y_X is a maximal set for which $\langle g \rangle Y_X \sqsubseteq \langle H \rangle^{\ast} X$ holds.

Take $F = \bigsqcap \{(X \times Y_X) \sqcup (\overline{X} \times \top) \mid X \in \mathscr{T} \operatorname{Src} f\}$. But is F complete or co-complete, or if meet is taken on Rel, do we have $F \in \operatorname{up} f$?

??

Let g be principal. Let a be an atomic filter. Take (??not always possible) $Y_a \in \operatorname{up} \langle f \rangle^{\ast} a$ such that $\langle g \rangle Y_a \sqsubseteq \langle H \rangle a$

Let $F = \bigsqcup_{a \in \mathrm{atoms}} (a \times Y_a)$ - also co-complete

??

Let for each $b \in \operatorname{atoms} \langle f \rangle^{\ast} a$ define $Z_b = ??$

??

Let f be complete. Replace it with principal funcoid F, such that $\langle f \rangle^{\ast} \{x\} \sqsubseteq \langle F \rangle^{\ast} x \sqsubseteq Y_{\{x\}}$. Prove $g \circ F \sqsubseteq H$

??

Split F into a join of monovalued functions. This does not work because every function produces its own g.

??

$G \circ f \sqsubseteq H$; $G \circ \operatorname{Compl} f = \operatorname{Compl}(G \circ f) \sqsubseteq H$

??

$\langle G \circ f \rangle^{\ast} \mathcal{X} = \langle G \rangle \langle f \rangle^{\ast} \mathcal{X} = \langle G \rangle \bigsqcap \langle \langle f \rangle^{\ast} \rangle^{\ast} \operatorname{up} \mathcal{X} \sqsubseteq \bigsqcap \langle \langle G \circ f \rangle^{\ast} \rangle^{\ast} \operatorname{up} \mathcal{X} \sqsubseteq ?? \sqsubseteq \langle g \circ f \rangle^{\ast} \mathcal{X} \sqsubseteq \langle H \rangle \mathcal{X}$

??

Find maximal [FIXME: there is no maximal because composition is not distributive over arbitrary joins.] funcoids F and G such that $G \circ F \sqsubseteq H$, then prove they are principal (or (co-)complete)

??

Replace G with G^M mapping $x \mapsto \langle G \rangle^{\ast} \{x\}$, Then ?? consider it as an isomorphism between sets.

$(Q \circ P)^M x = \langle Q \circ P \rangle^{\ast} \{x\} = \langle Q \rangle \langle P \rangle^{\ast} \{x\} = \langle Q \rangle P^M x$

$g \circ F \sqsubseteq H \Leftrightarrow \langle g \rangle \circ F^M \sqsubseteq H^M$

??

$f = (\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f$, $g = (\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} g$; $H \in \operatorname{up}(g \circ f)$

??

Use Todd Trimble's idea with ξ: $H \in \operatorname{up}(g \circ f) \Leftrightarrow (\mathcal{A} H^{\circledast} \mathcal{C} \Leftarrow \mathcal{A} (g \circ f)^{\circledast} \mathcal{C}) \Leftrightarrow (\mathcal{A} H^{\circledast} \mathcal{C} \Leftarrow \exists \mathcal{B}: (\mathcal{A} f^{\circledast} \mathcal{B} \wedge \mathcal{B} g^{\circledast} \mathcal{C})) \Leftrightarrow (A H^{\circledast} C \Leftarrow \exists B: (B \sqsupseteq \langle f \rangle A \wedge C \sqsupseteq \langle g \rangle B))$ [FIXME: check direction of implication]

$\{(A; C) \mid C \sqsupseteq \langle H \rangle A\} \supseteq \{(A; C) \mid \exists B: (B \sqsupseteq \langle f \rangle A \wedge C \sqsupseteq \langle g \rangle B)\}$.

Suppose $A \in \operatorname{dom} \{(A; C) \mid \exists B: (B \sqsupseteq \langle f \rangle A \wedge C \sqsupseteq \langle g \rangle B)\}$ or $C \in \operatorname{im} \{(A; C) \mid \exists B: (B \sqsupseteq \langle f \rangle A \wedge C \sqsupseteq \langle g \rangle B)\}$.

Then $\exists B: (B \sqsupseteq \langle f \rangle A \wedge C \sqsupseteq \langle g \rangle B)$. Take $B_{A,C}$ such that $B_{A,C} \sqsupseteq \langle f \rangle A \wedge C \sqsupseteq \langle g \rangle B_{A,C}$

Take $B'_A = \bigcap_{C \in ??} B_{A,C}$.

Then $B'_A \sqsupseteq \langle f \rangle A \wedge C \sqsupseteq \langle g \rangle B'_A$. [FIXME: it does not hold, only $B'_A \sqsupseteq \operatorname{Cor} \langle f \rangle A$]

Take co-complete funcoid $\langle F \rangle A = B'_A$. It is possible?? because $B'_{X \sqcup Y} = \bigcap_{C \in ??} B_{X \sqcup Y, C} = ??$

$B_{A, X \sqcup Y} \sqsupseteq \langle f \rangle (X \sqcup Y) \wedge C \sqsupseteq \langle g \rangle B_{X \sqcup Y, C}$

??

Instead of intersecting funcoids, consider join $f = \bigsqcup_{X \in ??} X \times Y_X$ or $f = \bigsqcup_{\mathcal{X} \in ??} \mathcal{X} \times Y_{\mathcal{X}}$. It is enough to consider ultrafilters $\mathcal{X}$. □

Theorem 13. *$\forall H \in \operatorname{up}(g \circ f) \exists F \in \operatorname{up} f, G \in \operatorname{up} g: H \sqsupseteq G \circ F$ for every composable funcoids f and g. [TODO: Also state it for reloids.]*

Proof. Let $H \in \mathrm{up}(g \circ f)$. Then $\exists F \in \mathrm{up}\, f: H \sqsupseteq g \circ F$ that is $H \in \mathrm{up}(g \circ F)$. Thus $\exists G \in \mathrm{up}\, g$ such that $H \sqsupseteq G \circ F$. □

3 Join of transitive reloids

$(f \sqcup g) \circ (f \sqcup g) \sqsupseteq (f \circ f) \sqcup (g \circ g)$ [need other direction]

Join of compositions of all finite sequences of f and g (in any order). It is equivalent to taking all alternating $S^*(f \circ g \circ f \circ \ldots \circ f)$ starting or ending with f or g.

"Relationships between completeness properties" in https://en.wikipedia.org/wiki/Completeness_%28order_theory%29

or alternatively:

$\mu = (f \sqcup g) \sqcup ((f \sqcup g) \circ (f \sqcup g)) \sqcup ((f \sqcup g) \circ (f \sqcup g) \circ (f \sqcup g)) \sqcup \ldots$

$\mu \circ \mu \sqsupseteq \mu$

"No similarly useful description of a subbase for the infimum of a family of quasi-uniformities is known." by http://www.sciencedirect.com/science/article/pii/S0166864107000181

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$\mathrm{up}\left(\bigsqcup_{n \in \mathbb{N}, f_{n,i} \in S} (f_{n,n} \circ \ldots \circ f_{n,0})\right) = \bigcap_{n \in \mathbb{N}, f_{n,i} \in S, F_{n,i} \in \mathrm{up}\, f_{n,i}} (F_{n,n} \circ \ldots \circ F_{n,0})$

Lemma 14. *$\mu = \bigsqcup_{n \in \mathbb{N}, f_{n,i} \in S} (f_{n,n} \circ \ldots \circ f_{n,0})$ is a transitive reloid, for every set S of endoreloids (on the same set).*

Proof. Denote $[U_n: n \in \mathbb{N}] = \bigsqcup_{n \in \mathbb{N}} (U_n \circ \ldots \circ U_0)$.

Let $U_{n,X} \in \mathrm{up}\, X$ for all $n \in \mathbb{N}$ and $X \in S$.

$\left[\bigcup_{X \in S} U_{n,X}: n \in \mathbb{N}\right]$ is ??

We need to prove $\mu \circ \mu \sqsubseteq \mu$.

$\left[\bigsqcup_{X \in S} U_{n,X}: n \in \mathbb{N}\right] \in \mathrm{up}\, \mu$.

Let $U_{n,X} \in X$ and $U_n = \bigcup_{X \in S} U_{n,X}$.

Then $\left[\bigsqcup_{X \in S} U_{2n,X}: n \in \mathbb{N}\right] \circ \left[\bigsqcup_{X \in S} U_{2n-1,X}: n \in \mathbb{N}\right] \subseteq \left[\bigsqcup_{X \in S} U_{n,X}: n \in \mathbb{N}\right]$ because?? $U_{2n,X} \circ U_{2n-1,X} \in \left[\bigsqcup_{X \in S} U_{n,X}: n \in \mathbb{N}\right]$?? [TODO: No need for reindexation.]

??

U_n is the join of all compositions of n-tuples. They form a generalized filter base. Thus it's enough to show that every U_n can be decomposed into smaller n-tuples. But that's obvious. (It isn't because it is an infinite join!)

Thus is above $U'_n \circ U''_n$.

Thus μ is also decomposed, because every its element is minorated as shown above.

??

Take $P \in \mathrm{up}\, \mu$. Then $P \in \mathrm{up}\, A$ for every $A \in \mathrm{up}(f_{n,n} \circ \ldots \circ f_{n,0})$.

$P \sqsupseteq F_{N,N} \circ \ldots \circ F_{N,0}$ for some $F_{N,i} \in \mathrm{up}\, f_{N,i}$ for all $N \in \mathbb{N}$.

Thus $P \sqsupseteq \bigsqcup_{n \in \mathbb{N}, f_i \in S} (F_{N,N} \circ \ldots \circ F_{N,0})$ where $F_{N,i} \in \mathrm{up}\, f_{N,i}$.

Take $F'_N = F_{N,\lfloor N/2 \rfloor} \circ \ldots \circ F_{N,0}$ and $F''_N = F_{N,N} \circ \ldots \circ F_{N,\lfloor N/2 \rfloor + 1}$

This way we exhaust all possible values?

$(F_{n,n} \circ \ldots \circ F_{n,n}) \circ (F_{m,m} \circ \ldots \circ F_{m,0}) \sqsubseteq P$. But this is obvious.

Thus easily follows $P \sqsupseteq \left(\bigsqcup_{n \in \mathbb{N}, f_i \in S} (F_{n,n} \circ \ldots \circ F_{n,0})\right) \circ \left(\bigsqcup_{n \in \mathbb{N}, f_i \in S} (F_{n,n} \circ \ldots \circ F_{n,0})\right)$; $P \in \mathrm{up}(\mu \circ \mu)$. □

Alternative formula: $\bigcup \langle \mathrm{GR} \rangle^* S \sqcup Z(\bigcup \langle \mathrm{GR} \rangle^* S) \sqcup Z(Z(\bigcup \langle \mathrm{GR} \rangle^* S)) \sqcup \ldots$

[TODO: Both for reloids and for Cauchy spaces Γ_i as in the attached article in email.]

[TODO: Also for funcoids (using (FCD)).]

3.1 Exponentials in category of graphs

http://arxiv.org/pdf/math/0605275.pdf definition 2.3 defines exponential graph

Let G and H are graphs.

The exponential graph $\mathrm{MOR}(G; H)$ is defined by the formulas:

$\mathrm{Ob}\,\mathrm{MOR}(G; H) = (\mathrm{Ob}\,H)^{\mathrm{Ob}\,G}$;

$(f; g) \in \mathrm{GR}\,\mathrm{MOR}(G; H) \Leftrightarrow \forall (v; w) \in \mathrm{GR}\,G: (f(v); g(w)) \in \mathrm{GR}\,H$

for every $f, g \in \mathrm{Ob}\,\mathrm{MOR}(G; H) = (\mathrm{Ob}\,H)^{\mathrm{Ob}\,G}$.

Equivalently

$(f; g) \in \mathrm{GR}\,\mathrm{MOR}(G; H) \Leftrightarrow \forall (v; w) \in \mathrm{GR}\,G: g \circ \{(v; w)\} \circ f^{-1} \subseteq \mathrm{GR}\,H$

$(f; g) \in \mathrm{GR}\,\mathrm{MOR}(G; H) \Leftrightarrow g \circ (\mathrm{GR}\,G) \circ f^{-1} \subseteq \mathrm{GR}\,H$

$(f; g) \in \mathrm{GR}\,\mathrm{MOR}(G; H) \Leftrightarrow \left\langle f \times^{(C)} g \right\rangle \mathrm{GR}\,G \subseteq \mathrm{GR}\,H$

Evaluation $\varepsilon: (\mathrm{MOR}(G; H) \times G) \rightarrow H$

If $(f; g) \in \mathrm{GR}\,\mathrm{MOR}(G; H)$ and $x \in \mathrm{GR}\,G$ then $\varepsilon((f; g); x) = (fx; gx) = g \circ \{(x; x)\} \circ f^{-1}$

3.2 Exponentials in category Fcd

The below gives definitions for exponential object, (exponential) evaluation, and exponential transpose, but no proof is given that they are really exponential object, (exponential) evaluation, and exponential transpose. Please write `porton@narod.ru` if you find a proof.

If G, H are endofuncoids, then $\mathrm{MOR}(G; H)$ (*exponential object*) is an endofuncoid.

$$\mathrm{MOR}(G; H) = \bigsqcup \left\{ t \in \operatorname{atoms} \mathsf{FCD}((\mathrm{Ob}\,H)^{\mathrm{Ob}\,G}; (\mathrm{Ob}\,H)^{\mathrm{Ob}\,G}) \mid (\operatorname{im} t) \circ G \circ (\operatorname{dom} t)^{-1} \sqsubseteq H \right\} = \bigsqcup \left\{ t \in \operatorname{atoms} \mathsf{FCD}((\mathrm{Ob}\,H)^{\mathrm{Ob}\,G}; (\mathrm{Ob}\,H)^{\mathrm{Ob}\,G}) \mid \left\langle (\operatorname{dom} t) \times^{(C)} (\operatorname{im} t) \right\rangle G \sqsubseteq H \right\}.$$

Evaluation:

$$\varepsilon\left(\left(f \times^{(A)} g \right) \times^{(A)} x \right) = \langle f \rangle (\mathsf{RLD})_{\mathrm{in}} x \times^{\mathsf{FCD}} \langle g \rangle (\mathsf{RLD})_{\mathrm{in}} x.$$

$$\varepsilon\left(\left(f \times^{(A)} g \right) \times^{(A)} x \right) = g \circ ((\mathsf{RLD})_{\mathrm{in}} x \times^{\mathsf{FCD}} (\mathsf{RLD})_{\mathrm{in}} x) \circ f^{-1} = \left\langle f \times^{(C)} g \right\rangle ((\mathsf{RLD})_{\mathrm{in}} x \times^{\mathsf{FCD}} (\mathsf{RLD})_{\mathrm{in}} x)$$

for atomic $f, g \in \mathsf{FCD}(\mathrm{Ob}\,G; \mathrm{Ob}\,H)$.

Evaluation $\varepsilon: \mathrm{MOR}(A; B) \times A \rightarrow B$

$$\varepsilon(F \times x) = \bigsqcup \left\{ g \circ (x \times^{\mathsf{FCD}} x) \circ f^{-1} \mid f, g \in \operatorname{atoms}^{\mathsf{FCD}(\mathrm{Ob}\,A; \mathrm{Ob}\,B)}, f \times g \not\asymp F \right\}.$$

$\varepsilon(F \times x) = \bigsqcup \{ g_1 \times^{\mathsf{FCD}} f_1 \mid f_0, g_0, f_1, g_1 \in \operatorname{atoms}^{\mathfrak{F}}, (f_0 \times^{\mathsf{FCD}} f_1) \times (g_0 \times^{\mathsf{FCD}} g_1) \not\asymp F \}$.

Proposition 15. $\varepsilon: \mathrm{MOR}(A; B) \times A \rightarrow B$.

Proof. We need to prove $\varepsilon \circ (\mathrm{MOR}(A; B) \times A) \sqsubseteq B \circ \varepsilon$. ?? □

Transpose $\sim f: Z \rightarrow \mathrm{MOR}(A; B)$ for a morphism $f: Z \times A \rightarrow B$

$$(\sim f)x = \bigsqcup \{ b \times^{\mathsf{FCD}} \langle f \rangle (x \times b) \mid b \in \operatorname{atoms}^{\mathsf{FCD}} \}.$$

Proposition 16. $\sim f: Z \rightarrow \mathrm{MOR}(A; B)$.

Proof. We need to prove $\sim f \circ Z \sqsubseteq \mathrm{MOR}(A; B) \circ \sim f$ whenever $f \circ (Z \times A) \sqsubseteq B \circ f$. ?? □

Awoday 6.5 "Equational definition" gives a simple way to check cartesian closed categories.

It's enough to prove:

1. $\varepsilon \circ (\sim f \times 1_A) = f$
2. $\varepsilon \circ \sim (g \times 1_A) = g$

$\varepsilon(\sim f \times 1_A)x = \varepsilon((\sim f)x \times x) = \bigsqcup \{ g_1 \times^{\mathsf{FCD}} f_1 \mid f_0, g_0, f_1, g_1 \in \operatorname{atoms}^{\mathfrak{F}}, (f_0 \times^{\mathsf{FCD}} f_1) \times (g_0 \times^{\mathsf{FCD}} g_1) \not\asymp (\sim f)x \} = ??$

??

4 Exponentials in category Rld

If now G, H are endoreloids, then $\mathrm{MOR}(G;H)$ is an endoreloid.

Definition 17. $\mathrm{MOR}_\alpha(G;H)=\bigsqcup\left\{t\in\mathrm{atoms}^{\mathsf{RLD}((\mathrm{Ob}\,H)^{\mathrm{Ob}\,G};(\mathrm{Ob}\,H)^{\mathrm{Ob}\,G})}\mid(\mathrm{im}\,t)\circ G\circ(\mathrm{dom}\,t)^{-1}\sqsubseteq H\right\}$.

Proposition 18. $\mathrm{MOR}_\alpha(G;H)=\bigsqcup\left\{t_0\times^{\mathsf{RLD}}t_1\mid t_0,t_1\in\mathrm{atoms}^{\mathfrak{F}((\mathrm{Ob}\,H)^{\mathrm{Ob}\,G})},t_1\circ G\circ t_0{}^{-1}\sqsubseteq H\right\}$.

Proof. If $t_0,t_1\in\mathfrak{F}((\mathrm{Ob}\,H)^{\mathrm{Ob}\,G})$ and $t_1\circ G\circ t_0{}^{-1}\sqsubseteq H$ then there are $t\in\mathrm{atoms}(t_0\times^{\mathsf{RLD}}t_1)$. Thus $(\mathrm{im}\,t)\circ G\circ(\mathrm{dom}\,t)^{-1}\sqsubseteq H$ and $t\in\mathrm{atoms}^{\mathsf{RLD}((\mathrm{Ob}\,H)^{\mathrm{Ob}\,G};(\mathrm{Ob}\,H)^{\mathrm{Ob}\,G})}$.

If $t\in\mathrm{atoms}^{\mathsf{RLD}((\mathrm{Ob}\,H)^{\mathrm{Ob}\,G};(\mathrm{Ob}\,H)^{\mathrm{Ob}\,G})}$ and $(\mathrm{im}\,t)\circ G\circ(\mathrm{dom}\,t)^{-1}\sqsubseteq H$ then $\mathrm{dom}\,t=t_0$ and $\mathrm{im}\,t=t_1$ for some $t_0,t_1\in\mathrm{atoms}^{\mathfrak{F}((\mathrm{Ob}\,H)^{\mathrm{Ob}\,G})}$ and thus $t_1\circ G\circ t_0{}^{-1}\sqsubseteq H$. □

Definition 19. $\mathrm{MOR}_\beta(G;H)=\bigsqcup\{t\in\mathsf{RLD}((\mathrm{Ob}\,H)^{\mathrm{Ob}\,G};(\mathrm{Ob}\,H)^{\mathrm{Ob}\,G})\mid(\mathrm{im}\,t)\circ G\circ(\mathrm{dom}\,t)^{-1}\sqsubseteq H\}$.

Conjecture 20. $\mathrm{MOR}_\alpha(G;H)=\mathrm{MOR}_\beta(G;H)$.

Obvious 21.

1. $\mathrm{MOR}_\alpha(G;H)=\bigsqcup\left\{t\in\mathrm{atoms}^{\mathsf{RLD}((\mathrm{Ob}\,H)^{\mathrm{Ob}\,G};(\mathrm{Ob}\,H)^{\mathrm{Ob}\,G})}\mid\left\langle(\mathrm{dom}\,t)\times^{(C)}(\mathrm{im}\,t)\right\rangle G\sqsubseteq H\right\}$;
2. $\mathrm{MOR}_\beta(G;H)=\bigsqcup\left\{t\in\mathsf{RLD}((\mathrm{Ob}\,H)^{\mathrm{Ob}\,G};(\mathrm{Ob}\,H)^{\mathrm{Ob}\,G})\mid\left\langle(\mathrm{dom}\,t)\times^{(C)}(\mathrm{im}\,t)\right\rangle G\sqsubseteq H\right\}$.

Conjecture 22.

1. $t\in\mathrm{atoms}\,\mathrm{MOR}_a(G;H)\Leftrightarrow(\mathrm{im}\,t)\circ G\circ(\mathrm{dom}\,t)^{-1}\sqsubseteq H$;
2. $t\in\mathrm{atoms}\,\mathrm{MOR}_\beta(G;H)\Leftrightarrow(\mathrm{im}\,t)\circ G\circ(\mathrm{dom}\,t)^{-1}\sqsubseteq H$.

Evaluation $\varepsilon\colon(\mathrm{MOR}(G;H)\times G)\to H$ that is for $x\in G=\mathsf{RLD}(\mathrm{Ob}\,G;\mathrm{Ob}\,G)$
it is a function on objects defined by the formula:
[TODO: Infer x and y from $x\times y$.]

$$\varepsilon((f\times^{\mathsf{RLD}}g)\times x)=g\circ(x\times^{\mathsf{RLD}}x)\circ f^{-1}=\left\langle f\times^{(C)}g\right\rangle(x\times^{\mathsf{RLD}}x)$$

for atomic $f,g\in\mathfrak{F}((\mathrm{Ob}\,H)^{\mathrm{Ob}\,G})$.

$\varepsilon(F\times x)=\bigsqcup\{g\circ(x\times^{\mathsf{RLD}}x)\circ f^{-1}\mid f,g\in\mathfrak{F}((\mathrm{Ob}\,H)^{\mathrm{Ob}\,G}),f\times^{\mathsf{RLD}}g\not\asymp F\}$
Let now $f\colon Z\times A\to B$.
In Set $\tilde{f}(a)(b)=f(a;b)$; $\tilde{f}(a)=b\mapsto f(a;b)$
$(\sim f)a=\bigsqcup\{b\times\langle(\mathsf{FCD})f\rangle(a\times b)\mid b\in\mathrm{atoms}^{\mathfrak{F}}\}$ [FIXME: Loss of information, see below.]
Awoday 6.5 "Equational definition" gives a simple way to check cartesian closed categories.
It's enough to prove:

1. $\varepsilon\circ(\sim f\times 1_A)=f$
2. $\varepsilon\circ\sim(g\times 1_A)=g$

Really, $\varepsilon(\sim f\times 1_A)z=\varepsilon(\sim fz\times z)=\varepsilon(\bigsqcup\{b\times\langle(\mathsf{FCD})f\rangle(z\times b)\mid b\in\mathrm{atoms}^{\mathfrak{F}}\}\times z)=\bigsqcup\{g'\circ(z\times^{\mathsf{RLD}}z)\circ f'^{-1}\mid f',g'\in\mathfrak{F}((\mathrm{Ob}\,H)^{\mathrm{Ob}\,G}),f'\times^{\mathsf{RLD}}g'\not\asymp\bigsqcup\{b\times\langle(\mathsf{FCD})f\rangle(z\times b)\mid b\in\mathrm{atoms}^{\mathfrak{F}}\}\}$
[FIXME: It cannot be equal to f due loss on information in (FCD).]

5 On decomposisiton of binary relations and reloids

Example 23. $\rho\bigsqcap G\neq\bigsqcap\langle\rho\rangle G$ for some st G of reloids (with matching sources and destinations).

Proof. Take $\Delta=\bigsqcap\{\uparrow^{\mathbb{R}}(-\varepsilon;\varepsilon)\mid\varepsilon>0\}$. Take $\{\alpha\}\times^{\mathsf{RLD}}p$ where $p\sqsubseteq\Delta$ is a nontrivial ultrafilter.

$\langle\rho\bigsqcap G\rangle(\{\alpha\}\times^{\mathsf{RLD}}p)=(\bigsqcap G)\circ(\{\alpha\}\times^{\mathsf{RLD}}p)$

$\langle\bigsqcap\langle\rho\rangle G\rangle(\{\alpha\}\times^{\mathsf{RLD}}p)=$ (because $\{\alpha\}\times^{\mathsf{RLD}}p$ is atomic)$=\bigsqcap\{\langle\rho g\rangle(\{\alpha\}\times^{\mathsf{RLD}}p)\mid g\in G\}=\bigsqcap\{g\circ(\{\alpha\}\times^{\mathsf{RLD}}p)\mid g\in G\}$. □

6 Rest

Lemma 24. *Every non-empty set has a well ordering with greatest element.*

Proof. Take an arbitrary well ordering of our set and move the first element to the end of the order. □

Theorem 25. $L\in[f]\Rightarrow[f]\cap\prod_{i\in\mathrm{dom}\,\mathfrak{A}}\mathrm{atoms}\,L_i\neq\emptyset$ *for every pre-multifuncoid* f *of the form whose elements are atomic posets.*

Proof. If arity $f=0$ our theorem is trivial, so let arity $f\neq 0$. Let $\sqsubseteq$ is a well-ordering of arity f with greatest element m (it exists by the lemma).

Let Φ is a function which maps non-least elements of posets into atoms under these elements and least elements into themselves. (Note that Φ is defined on least elements only for completeness, Φ is never taken on a least element in the proof below.) [TODO: Fix the “universal set” paradox here.]

Define a transfinite sequence a by transfinite induction with the formula

$$a_c=\Phi\langle f\rangle_c\,(a|_{X(c)\setminus\{c\}}\cup L|_{(\mathrm{arity}\,f)\setminus X(c)}).$$

Let $b_c=a|_{X(c)\setminus\{c\}}\cup L|_{(\mathrm{arity}\,f)\setminus X(c)}$. Then $a_c=\Phi\langle f\rangle_c\,b_c$.

Let us prove by transfinite induction

$$a_c\in\mathrm{atoms}\,L_c.$$

$a_c=\Phi\langle f\rangle_c\,L|_{(\mathrm{arity}\,f)\setminus\{c\}}\sqsubseteq\langle f\rangle_c\,L|_{(\mathrm{arity}\,f)\setminus\{c\}}$. Thus $a_c\sqsubseteq L_c$. [TODO: Is it true for *pre*-multifuncoids?]

The only thing remained to prove is that $\langle f\rangle_c\,b_c\neq 0$

that is $\langle f\rangle_c\,(a|_{X(c)\setminus\{c\}}\cup L|_{(\mathrm{arity}\,f)\setminus X(c)})\neq 0$ that is $y\not\asymp\langle f\rangle_c\,b_c$

??

$L_c\not\asymp\langle f\rangle_c\;b_c\Leftrightarrow b_c(0)\not\asymp\langle f\rangle_0(b_c|_{(\mathrm{arity}\,f)\setminus\{0\}}\cup\{c;\;L_c\})\Leftrightarrow a_0\not\asymp\langle f\rangle_0(a|_{X(c)\setminus\{0,c\}}\cup L|_{(\mathrm{arity}\,f)\setminus X(c)}\cup\{c;L_c\})\Leftrightarrow a_0\not\asymp\langle f\rangle_0(a|_{X(c)\setminus\{0,c\}}\cup L|_{((\mathrm{arity}\,f)\setminus X(c))\cup\{c\}})\Leftarrow a_0\not\asymp\langle f\rangle_0(a|_{X(c)\setminus\{0,c\}}\cup a|_{((\mathrm{arity}\,f)\setminus X(c))\cup\{c\}})\Leftrightarrow a_0\not\asymp\langle f\rangle_0(a|_{(X(c)\setminus\{0,c\})\cup((\mathrm{arity}\,f)\setminus X(c))\cup\{c\}})\Leftrightarrow a_0\not\asymp\langle f\rangle_0(a|_{(\mathrm{arity}\,f)\setminus\{0\}})\Leftrightarrow\Phi\langle f\rangle_0\,(L|_{(\mathrm{arity}\,f)\setminus\{0\}})\not\asymp\langle f\rangle_0(a|_{(\mathrm{arity}\,f)\setminus\{0\}})$.

??

$a_0=\Phi\langle f\rangle_0\,(L|_{(\mathrm{arity}\,f)\setminus\{0\}})$

??

??Two ways to prove:??

$L_c\not\asymp\langle f\rangle_c\;b_c\Leftrightarrow b_c(k)\not\asymp\langle f\rangle_k(b_c|_{(\mathrm{arity}\,f)\setminus\{k\}}\cup\{c;\;L_c\})\Leftrightarrow a_k\not\asymp\langle f\rangle_k(a|_{X(c)\setminus\{k,c\}}\cup L|_{(\mathrm{arity}\,f)\setminus X(c)}\cup\{c;L_c\})\Leftrightarrow a_k\not\asymp\langle f\rangle_k(a|_{X(c)\setminus\{k,c\}}\cup L|_{((\mathrm{arity}\,f)\setminus X(c))\cup\{c\}})\Leftarrow a_k\not\asymp\langle f\rangle_k(a|_{X(c)\setminus\{k,c\}}\cup a|_{((\mathrm{arity}\,f)\setminus X(c))\cup\{c\}})\Leftrightarrow a_k\not\asymp\langle f\rangle_k(a|_{(X(c)\setminus\{k,c\})\cup((\mathrm{arity}\,f)\setminus X(c))\cup\{c\}})\Leftrightarrow a_k\not\asymp\langle f\rangle_k(a|_{(\mathrm{arity}\,f)\setminus\{k\}})$

??

$L_c\not\asymp\langle f\rangle_c\,b_c\Leftrightarrow a_k\not\asymp\langle f\rangle_k\,(a|_{X(c)\setminus\{c,k\}}\cup L|_{(\mathrm{arity}\,f)\setminus X(c)}\cup\{(c;\,L_c)\})\Leftrightarrow a_k\not\asymp\langle f\rangle_k\,(a|_{X(c)\setminus\{c,k\}}\cup L|_{((\mathrm{arity}\,f)\setminus X(c))\cup\{c\}})\Leftarrow a_k\not\asymp\langle f\rangle_k\,(a|_{X(c)\setminus\{c,k\}}\cup a|_{((\mathrm{arity}\,f)\setminus X(c))\cup\{c\}})\Leftrightarrow a_k\not\asymp\langle f\rangle_k\,(a|_{(\mathrm{arity}\,f)\setminus\{k\}})$

??

$y\not\asymp\langle f\rangle_c\;b_c\Leftrightarrow a_k\not\asymp\langle f\rangle_k\;(a|_{X(c)\setminus\{c,k\}}\cup L|_{(\mathrm{arity}\,f)\setminus X(c)}\cup\{(c;\;y)\})\Leftarrow a_k\not\asymp\langle f\rangle_k\;(a|_{X(k)\setminus\{k\}}\cup L|_{(\mathrm{arity}\,f)\setminus X(c)}\cup\{(c;y)\})$

$y\not\asymp\langle f\rangle_c\,b_c\Leftrightarrow L_k\not\asymp\langle f\rangle_k\,(a|_{X(c)\setminus\{c\}}\cup L|_{(\mathrm{arity}\,f)\setminus(X(c)\cup\{k\})}\cup\{(c;y)\})\Leftarrow$

??

$a_m=\Phi\langle f\rangle_m(\lambda i\in(\mathrm{arity}\;f)\setminus\{m\}\colon a_i)=\Phi\langle f\rangle_m\;a|_{(\mathrm{arity}\,f)\setminus\{m\}}$; $a_m\not\asymp\Phi\langle f\rangle_m\;a|_{(\mathrm{arity}\,f)\setminus\{m\}}$; $a_m\not\asymp\langle f\rangle_m\,a|_{(\mathrm{arity}\,f)\setminus\{m\}}$; $a\in[f]$. □

Theorem 26. *Let* $\mathfrak{A}=\mathfrak{A}_{i\in n}$ *is a family of boolean lattices.*

A relation $\delta\in\mathscr{P}\prod\mathrm{atoms}^{\mathfrak{F}(\mathfrak{A}_i)}$ *such that for every* $a\in\prod\mathrm{atoms}^{\mathfrak{F}(\mathfrak{A}_i)}$

$$\forall A\in a\colon\delta\cap\prod_{i\in n}\mathrm{atoms}\,{\uparrow}^{\mathfrak{A}_i}A_i\neq\emptyset\Rightarrow a\in\delta\qquad(3)$$

can be continued till the function $\upuparrows f$ *for a unique staroid* f *of the form* $\lambda i\in n\colon\mathfrak{P}(\mathfrak{A}_i)$. *The funcoid* f *is completary.*

For every $\mathcal{X} \in \prod_{i\in n} \mathfrak{F}(\mathfrak{A}_i)$

$$\mathcal{X} \in \mathrm{GR} \upuparrows f \Leftrightarrow \delta \cap \prod_{i\in n} \mathrm{atoms}\, \mathcal{X}_i \neq \emptyset. \tag{4}$$

Proof. [FIXME: ??Use of unproved conjecture ?.]

By the theorem 11 (used that it is a boolean lattice) we have $\mathcal{X} \in \mathrm{GR} \upuparrows f \Leftrightarrow \mathrm{GR} \upuparrows f \cap \prod_{i\in n} \mathrm{atoms}\, \mathcal{X}_i \neq \emptyset$ and thus (4). From this also follows uniqueness.

It is left to prove that there exists a completary staroid f such that $\upuparrows f$ is a continuation of δ.

Consider the relation f defined by the formula $X \in f \Leftrightarrow \delta \cap \prod_{i\in n} \mathrm{atoms} \uparrow^{\mathfrak{A}_i} X_i \neq \emptyset$.

$I_0 \sqcup I_1 \in f \Leftrightarrow \delta \cap \prod_{i\in n} \mathrm{atoms} \uparrow^{\mathfrak{A}_i} (I_0\, i \sqcup I_1\, i) \neq \emptyset \Leftrightarrow \delta \cap \prod_{i\in n} (\mathrm{atoms} \uparrow^{\mathfrak{A}_i} I_0\, i \cup \mathrm{atoms} \uparrow^{\mathfrak{A}_i} I_1\, i) \neq \emptyset$.

Thus by the lemma $I_0 \sqcup I_1 \in f \Leftrightarrow \exists c \in \{0, 1\}^n \colon \delta \cap \prod_{i\in n} \mathrm{atoms} \uparrow^{\mathfrak{A}_i} I_{c(i)} \neq \emptyset \Leftrightarrow \exists c \in \{0, 1\}^n \colon (\lambda i \in n \colon I_{c(i)} i) \in f$. Trivially if $\exists i \in n \colon X_i = 0$ then $X \notin f$. So f is a completary staroid.

Let $a \in \prod \mathrm{atoms}^{\mathfrak{F}(\mathfrak{A}_i)}$.

The reverse of (3) is obvious. So we have $a \in \delta \Leftrightarrow \forall A \in a \colon \delta \cap \prod_{i\in n} \mathrm{atoms} \uparrow^{\mathfrak{A}_i} A_i \neq \emptyset \Leftrightarrow \forall A \in a \colon A \in f \Leftrightarrow \forall A \in a \colon A \in f \Leftrightarrow a \subseteq f \Leftrightarrow a \in \upuparrows f$. Thus $\upuparrows f$ is a continuation of δ. □

$\langle f_0 \times f_1 \rangle x = \bigsqcup \{ \langle f_0 \rangle X \times^{\mathsf{FCD}} \langle f_1 \rangle X \mid X \in \mathrm{atoms}\, x \}$

$\langle (f_0 \times f_1)^{-1} \rangle y = \langle f_0{}^{-1} \rangle \mathrm{dom}\, y \sqcap \langle f_1{}^{-1} \rangle \mathrm{im}\, y$ (not only for atomic y):

$\langle (f_0 \times f_1)^{-1} \rangle y = \bigsqcup \{ \langle (f_0 \times f_1)^{-1} \rangle \{p\} \mid p \in y \} = \bigsqcup \{ \langle f_0{}^{-1} \rangle \mathrm{dom}\, \{p\} \sqcap \langle f_1{}^{-1} \rangle \mathrm{im}\, \{p\} \mid p \in y \} = \bigsqcup \{ \langle f_0{}^{-1} \rangle \mathrm{dom}\, \{p\} \mid p \in y \} \sqcap \bigsqcup \{ \langle f_1{}^{-1} \rangle \mathrm{im}\, \{p\} \mid p \in y \} = \langle f_0{}^{-1} \rangle \mathrm{dom}\, y \sqcap \langle f_1{}^{-1} \rangle \mathrm{im}\, y$

It seems that these are not components of a funcoid.

Conjecture 27. *Let R is a set of staroids of the form $\lambda i \in n \colon \mathfrak{F}(\mathfrak{A}_i)$ where every $\mathfrak{A}_i$ is a boolean lattice. If $x \in \prod_{i\in n} \mathrm{atoms}^{\mathfrak{F}(\mathfrak{A}_i)}$ then $x \in \mathrm{GR} \upuparrows \bigsqcap R \Leftrightarrow \forall f \in R \colon x \in \upuparrows f$.*

Proof. Let denote $x \in \delta \Leftrightarrow \forall f \in R \colon x \in \upuparrows f$ for every $x \in \prod_{i\in n} \mathrm{atoms}^{\mathfrak{F}(\mathfrak{A}_i)}$. For every $a \in \prod_{i\in n} \mathrm{atoms}^{\mathfrak{F}(\mathfrak{A}_i)}$

$\forall X \in a \colon \delta \cap \prod_{i\in n} \mathrm{atoms} \uparrow^{\mathfrak{A}_i} X_i \neq \emptyset \Leftrightarrow \forall X \in a \exists x \in \prod_{i\in n} \mathrm{atoms} \uparrow^{\mathfrak{A}_i} X_i \colon x \in \delta \Leftrightarrow \forall X \in a \exists x \in \prod_{i\in n} \mathrm{atoms} \uparrow^{\mathfrak{A}_i} X_i \forall f \in R \colon x \in \upuparrows f \Rightarrow \forall X \in a, f \in R \exists x \in \prod_{i\in n} \mathrm{atoms} \uparrow^{\mathfrak{A}_i} X_i \colon x \in \upuparrows f \Rightarrow \forall X \in a, f \in R \colon X \in f \Leftrightarrow \forall f \in R \colon a \subseteq f \Leftrightarrow \forall f \in R \colon a \in \upuparrows f \Leftrightarrow a \in \delta$.

So by the previous theorem δ can be contimued till $\upuparrows p$ for some staroid p of the form $\lambda i \in n \colon \mathfrak{P}(\mho_i)$.

Let's prove $p = \bigsqcap R$.

$x \in \upuparrows p \Leftrightarrow x \in \delta \Rightarrow x \in \upuparrows f$ for every $f \in R$ and $x \in \prod_{i\in n} \mathrm{atoms}^{\mathfrak{F}(\mathfrak{A}_i)}$. Thus $\upuparrows p \subseteq \upuparrows f$. Consequently $\forall f \in R \colon p \subseteq f$.

Suppose that q is a staroid of the form $\lambda i \in n \colon \mathfrak{P}(\mathfrak{A}_i)$ such that $\forall f \in R \colon q \subseteq f$. Then for every $x \in \prod_{i\in n} \mathrm{atoms}^{\mathfrak{F}(\mathfrak{A}_i)}$ we have $x \in \upuparrows q \Rightarrow \forall f \in R \colon x \in \upuparrows f \Leftrightarrow x \in \delta \Leftrightarrow x \in \upuparrows p$. So $\upuparrows q \subseteq \upuparrows p$ that is $q \subseteq p$.

We have proved $p = \bigsqcap R$. It's remained to prove that $x \in \upuparrows p \Leftrightarrow \forall f \in R \colon x \in \upuparrows f$ for every $x \in \prod_{i\in n} \mathrm{atoms}^{\mathfrak{F}(\mathfrak{A}_i)}$. Really, $x \in \upuparrows p \Leftrightarrow x \in \delta \Leftrightarrow \forall f \in R \colon x \in \upuparrows f$. □

Example 28. There exists a multifuncoid on power sets, which is not a completary multifuncoid.

Proof. Let $\mathrm{arity}\, f = \mathbb{N}$ and $\mathrm{form}\, f = (\mathscr{P}\{0,1\})^{\mathbb{N}}$.

Characteristic function of a set D is $\Lambda(D) = \lambda i \in \mathbb{N} \colon \begin{cases} \{1\} & \text{if } i \in D; \\ \{0\} & \text{if } i \notin D. \end{cases}$

$\mathrm{GR}\, f = \{ \Lambda(D) \mid D \in \Omega \}$.

Obviously f is an anchored relation.

Let $k \in \mathbb{N}$.

$(\mathrm{val}\, f)_k L = \{ X \in (\mathrm{form}\, f)_k \mid L \cup \{(k; X)\} \in \mathrm{GR}\, f \} = \{ X \in (\mathscr{P}\{0\})^{\mathbb{N}} \mid \exists D \in \Omega \colon L \cup \{(k; X)\} = \Lambda(D) \} = \{ X \in (\mathscr{P}\{0,1\})^{\mathbb{N}} \mid \exists D \in \Omega \colon (L = \Lambda(D)|_{\mathbb{N} \setminus \{k\}} \wedge X = \Lambda(D)(k)) \}$.

$X \in (\mathrm{val}\, f)_k L \Leftrightarrow \exists D \in \Omega \colon (L = \Lambda(D)|_{\mathbb{N} \setminus \{k\}} \wedge X_k = \Lambda(D)(k)) \Leftrightarrow ?? \Leftrightarrow \exists D \in \Omega^{\mathbb{N} \setminus \{k\}}, P \in \{\{0\}, \{1\}\} \colon (L = \Lambda(D) \wedge X_k = P) \Leftrightarrow \exists D \in \Omega^{\mathbb{N} \setminus \{k\}} \colon L = \Lambda(D) \wedge \exists P \in \{\{0\}, \{0, 1\}\} \colon X_k = P \Leftrightarrow \exists D \in \Omega^{\mathbb{N} \setminus \{k\}} \colon L = \Lambda(D) \wedge X \in \{\{0\}, \{1\}\}$ (Note it does not depend on X.)

Let $X, Y \in (\mathscr{P}\{0,1\})^{\mathbb{N}}$. Then $X \sqcup Y \in (\mathrm{val}\, f)_k L \Leftrightarrow \exists D \in \Omega^{\mathbb{N} \setminus \{k\}} \colon L = \Lambda(D) \wedge X \sqcup Y \in \{\{0\}, \{1\}\}$

If $X\in\{\{0\}\}$, $Y\in\{\{1\}\}$ then $X\sqcup Y\notin\{\{0\},\{0,1\}\}$
??
That $X_i=\emptyset\Rightarrow X\notin(\mathrm{val}\,f)_k L$ is obvious. So f is a pre-multifuncoid.
?? □

Conjecture 29. *If a is a completary multifuncoid and $\mathrm{Dst}\,f_i$ is a starrish poset for every $i\in n$ then $\mathrm{StarComp}(a;f)$ is a completary multifuncoid.*

Proof. Let $\forall K\in\prod\mathrm{form}\,f\colon(K\sqsupseteq L_0\wedge K\sqsupseteq L_1\Rightarrow K\in\mathrm{StarComp}(a;f))$ that is $\forall K\in\prod\mathrm{form}\,f\colon\left(K\sqsupseteq L_0\wedge K\sqsupseteq L_1\Rightarrow\exists y\in\prod_{i\in n}\mathrm{atoms}\,\mathfrak{A}_i\colon(\forall i\in n\colon y_i\,[f_i]\,K_i\wedge y\in a)\right)$ that is $\forall K\in\prod\mathrm{form}\,f\exists y\in\prod_{i\in n}\mathrm{atoms}\,\mathfrak{A}_i\colon(K\sqsupseteq L_0\wedge K\sqsupseteq L_1\Rightarrow(\forall i\in n\colon y_i\,[f_i]\,K_i\wedge y\in a))$ that is $\forall K\in\prod\mathrm{form}\,f\exists y\in\prod_{i\in n}\mathrm{atoms}\,\mathfrak{A}_i\colon((K\sqsupseteq L_0\wedge K\sqsupseteq L_1\Rightarrow\forall i\in n\colon y_i\,[f_i]\,K_i)\wedge y\in a)$ that is?? $\forall K\in\prod\mathrm{form}\,f\exists y\in\prod_{i\in n}\mathrm{atoms}\,\mathfrak{A}_i\colon(\exists c\in\{0,1\}^n\forall i\in n\colon y_i\,[f_i]\,L_{c(i)}\,i\wedge y\in a)$ that is ?? □

Conjecture 30. *$\prod^{(D)}F$ is a pre-multifuncoid if every F_i is a pre-multifuncoid.*

Proof. Let $X,Y\in\left(\mathrm{form}\prod^{(D)}F\right)_{(i;j)}$.

$\forall Z\in\left(\mathrm{form}\prod^{(D)}F\right)_{(i;j)}\colon\left(Z\supseteq X\wedge Z\supseteq Y\Rightarrow Z\in\left(\mathrm{val}\prod^{(D)}F\right)_{(i;j)}L\right)\Leftrightarrow\forall Z\in\left(\mathrm{form}\prod^{(D)}F\right)_{(i;j)}\colon(Z\supseteq X\wedge Z\supseteq Y\Rightarrow\exists K\in(\mathrm{form}\,F_i)|_{(\mathrm{arity}\,F_i)\setminus\{j\}}\colon Z\in(\mathrm{val}\,F_j)K)\Leftrightarrow\forall Z\in\left(\mathrm{form}\prod^{(D)}F\right)_{(i;j)}\colon(\exists K\in(\mathrm{form}\,F_i)|_{(\mathrm{arity}\,F_i)\setminus\{j\}}\colon(Z\supseteq X\wedge Z\supseteq Y\Rightarrow Z\in(\mathrm{val}\,F_j)K))\Leftrightarrow$?? $\Leftrightarrow\forall Z\in\left(\mathrm{form}\prod^{(D)}F\right)_{(i;j)}\colon(\exists K\in(\mathrm{form}\,F_i)|_{(\mathrm{arity}\,F_i)\setminus\{j\}}\colon(X\in(\mathrm{val}\,F_j)K\vee Y\in(\mathrm{val}\,F_j)K))\Leftrightarrow\forall Z\in\left(\mathrm{form}\prod^{(D)}F\right)_{(i;j)}\colon(\exists K\in(\mathrm{form}\,F_i)|_{(\mathrm{arity}\,F_i)\setminus\{j\}}\colon X\in(\mathrm{val}\,F_j)K\vee\exists K\in(\mathrm{form}\,F_i)|_{(\mathrm{arity}\,F_i)\setminus\{j\}}\colon Y\in(\mathrm{val}\,F_j)K)\Leftrightarrow$ □

Let f is a funcoid.
Then there exists a reloid g such that ??
=============
$(\mathsf{RLD})_{\mathrm{in}}f=\bigcup\left\{a\times^{\mathsf{RLD}}b\mid a\in\mathrm{atoms}\,1^{\mathfrak{F}(\mathrm{Src}\,f)},b\in\mathrm{atoms}\,1^{\mathfrak{F}(\mathrm{Dst}\,f)},a\times^{\mathsf{FCD}}b\subseteq f\right\}$
$(\mathsf{RLD})_{\mathrm{in}}(g\circ f)=\bigcup\left\{a\times^{\mathsf{RLD}}b\mid a\in\mathrm{atoms}\,1^{\mathfrak{F}(\mathrm{Src}\,f)},b\in\mathrm{atoms}\,1^{\mathfrak{F}(\mathrm{Dst}\,f)},a\times^{\mathsf{FCD}}b\subseteq g\circ f\right\}=\bigcup\left\{a\times^{\mathsf{RLD}}b\mid a\in\mathrm{atoms}\,1^{\mathfrak{F}(\mathrm{Src}\,f)},b\in\mathrm{atoms}\,1^{\mathfrak{F}(\mathrm{Dst}\,f)},a\times^{\mathsf{RLD}}b\subseteq(\mathsf{RLD})_{\mathrm{in}}(g\circ f)\right\}$

$(\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD})((\mathsf{RLD})_{\mathrm{in}}g\circ(\mathsf{RLD})_{\mathrm{in}}f)=(\mathsf{RLD})_{\mathrm{in}}((\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}}g\circ(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}}f)=(\mathsf{RLD})_{\mathrm{in}}(g\circ f)$

Lemma 31. *$\forall Y\in\mathrm{up}\,\langle f\rangle^*X\exists F\in\mathrm{up}\,f\colon\langle F\rangle X\subseteq Y$ for every funcoid f.*

Proof. ?? □

$(\mathsf{RLD})_{\mathrm{in}}(g\circ f)=$

Theorem 32. *$g\circ f=\bigcap\left\{\uparrow^{\mathsf{FCD}(\mathrm{Src}\,f;\mathrm{Dst}\,g)}(G\circ F)\mid F\in\mathrm{up}\,f,G\in\mathrm{up}\,g\right\}$*

Proof. It's enough?? to prove that $\forall H\in\mathrm{up}(g\circ f)\exists F\in\mathrm{up}\,f,G\in\mathrm{up}\,g\colon H\supseteq G\circ F$.

$X\,[g\circ f]^*\,Y\Leftrightarrow\langle f\rangle^*X\not\asymp\langle g^{-1}\rangle^*Y\Leftrightarrow\forall X'\in\mathrm{up}\,\langle f\rangle^*X,Y'\in\mathrm{up}\,\langle g^{-1}\rangle^*Y\colon X'\not\asymp Y'\Leftrightarrow\exists F\in\mathrm{up}\,f,G\in\mathrm{up}\,g\colon\langle F\rangle X\not\asymp\langle G\rangle Y\Leftrightarrow\exists F\in\mathrm{up}\,f,G\in\mathrm{up}\,g\colon X\,[G\circ F]\,Y$ (used the lemma).

Let $H\in\mathrm{up}(g\circ f)$. Then $X\,[H]^*\,Y\Rightarrow X\,[g\circ f]^*\,Y\Rightarrow\exists F\in\mathrm{up}\,f,G\in\mathrm{up}\,g\colon X\,[G\circ F]\,Y$ for every X, Y. Thus ??(it does not work because F amd G depend on X and Y). □

Lemma 33. *$f\circ r=\bigcap\{f\circ\uparrow^{\mathsf{FCD}}R\mid R\in\mathrm{up}\,r\}$*

Proof. Obviously $f\circ r\subseteq\bigcap\{f\circ\uparrow^{\mathsf{FCD}}R\mid R\in\mathrm{up}\,r\}$.

$\langle f\circ r\rangle^*X\supseteq$??$\bigcap\{\langle f\rangle\langle R\rangle X\mid R\in\mathrm{up}\,r\}=\bigcap\{\langle f\rangle\langle\uparrow^{\mathsf{FCD}}R\rangle^*X\mid R\in\mathrm{up}\,r\}=\bigcap\{\langle f\circ\uparrow^{\mathsf{FCD}}R\rangle^*X\mid R\in\mathrm{up}\,r\}\supseteq\langle\bigcap\{f\circ\uparrow^{\mathsf{FCD}}R\mid R\in\mathrm{up}\,r\}\rangle^*X$

$W=\{\langle R\rangle X \mid R\in\operatorname{up} r\}$ is?? a generalized filter base?? (No, it isn't, $\operatorname{up} r$ isn't a filter.)
Let $P_0, P_1\in W$. Then $P_0=\langle R_0\rangle X$, $P_1=\langle R_1\rangle X$

Let $F\in\operatorname{up}(f\circ r)$. Then ?? $F\in\operatorname{up}\bigcap\{f\circ\uparrow^{\mathsf{FCD}}R \mid R\in\operatorname{up} r\}$.
So $f\circ r\supseteq\bigcap\{f\circ\uparrow^{\mathsf{FCD}}R \mid R\in\operatorname{up} r\}$

$\mathcal{X}\,[f\circ\uparrow^{\mathsf{FCD}}R]\,\mathcal{Y}\Leftrightarrow[\uparrow^{\mathsf{FCD}}R]\mathcal{X}\not\asymp\langle f^{-1}\rangle\mathcal{Y}$
??
Let $W=\{\langle\uparrow^{\mathsf{FCD}}R\rangle a \mid R\in\operatorname{up} r\}$. W is a generalized filter base??. Really If $\mathcal{X},\mathcal{Y}\in W$. Then $\mathcal{X}=\langle\uparrow^{\mathsf{FCD}}X\rangle a$ and $\mathcal{Y}=\langle\uparrow^{\mathsf{FCD}}Y\rangle a$ for some $X\in\operatorname{up} r$, $Y\in\operatorname{up} r$.

$$\begin{aligned}\langle\bigcap\{f\circ\uparrow^{\mathsf{FCD}}R \mid R\in\operatorname{up} r\}\rangle a &= \\ \bigcap\{\langle f\circ\uparrow^{\mathsf{FCD}}R\rangle a \mid R\in\operatorname{up} r\} &= (\text{because } \{\langle\uparrow^{\mathsf{FCD}}R\rangle a \mid R\in\operatorname{up} r\} \text{ is a g.f.b.??}) \\ \langle f\rangle\bigcap\{\langle\uparrow^{\mathsf{FCD}}R\rangle a \mid R\in\operatorname{up} r\} &= \\ \langle f\rangle\langle r\rangle a\end{aligned}$$

Counter-example attempt: Let $r=\mathrm{id}^{\mathsf{FCD}}_{\Omega}$. ?? □

Corollary 34. $g\circ f=\bigcap\left\{\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} g)}(G\circ F) \mid F\in\operatorname{up} f, G\in\operatorname{up} g\right\}$.

Proof. $x\left[\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} g)}(G\circ F)\right]z\Leftrightarrow\exists y\in\operatorname{atoms}\operatorname{Dst} f\colon(x\,[\uparrow F]\,y\wedge y\,[\uparrow G]\,z)$

$x\ \left[\bigcap\left\{\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} g)}(G\circ F) \mid F\in\operatorname{up} f, G\in\operatorname{up} g\right\}\right]\ z \Leftrightarrow \forall F\in\operatorname{up} f, G\in\operatorname{up} g\colon x\left[\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} g)}(G\circ F)\right]z\Leftrightarrow\forall F\in\operatorname{up} f, G\in\operatorname{up} g\exists y\in\operatorname{atoms}\operatorname{Dst} f\colon\left(x\left[\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} g)}F\right]y\wedge y\left[\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} g)}G\right]z\right)$ □

Conjecture 35. *Let $\mho$ be a set, $\mathfrak{F}$ be the set of f.o. on $\mho$, $\mathfrak{P}$ be the set of principal f.o. on $\mho$, let n be an index set. Consider the filtrator $(\mathfrak{F}^n;\mathfrak{P}^n)$. Then if f is a multifuncoid of the form $\mathfrak{P}^n$, then E^*f is a multifuncoid of the form $\mathfrak{F}^n$.*

Proof. $(\operatorname{val} E^*f)_i L=\,??=\{X\in\mathfrak{A}_i \mid \forall K\in\operatorname{up} L\colon K\cup\{(i;X)\}\in f\}=\{X\in\mathfrak{A}_i \mid \forall K\in\operatorname{up} L\colon X\in(\operatorname{val} f)_i K\}=\{X\in\mathfrak{A}_i \mid \forall K\in\operatorname{up} L\colon K\cup\{(i;X)\}\in f\}=\,??=\{X\in\mathfrak{A}_i \mid L\cup\{(i;X)\}\in E^*f\}$

$X\in(\operatorname{val} E^*f)_i L\Leftrightarrow\,??\Leftrightarrow L\cup\{(i;X)\}\in f$

$A\cup B\in(\operatorname{val} E^*f)_i L=\,??=L\cup\{(i;A\cup B)\}\in f\Leftrightarrow$ (futher trivial)

??

$(\operatorname{val} E^*f)_i L=\{X\in\mathfrak{A}_i \mid L\cup\{(i;X)\}\in E^*f\}=\{X\in\mathfrak{A}_i \mid \operatorname{up}(L\cup\{(i;X)\})\subseteq f\}=\,??=\{X\in\mathfrak{A}_i \mid \operatorname{up} L\times X\subseteq f\}=\{X\in\mathfrak{A}_i \mid \forall K\in\operatorname{up} L, x\in\operatorname{up} X\colon K\cup\{(i;x)\}\in f\}\Leftrightarrow\{X\in\mathfrak{A}_i \mid \forall K\in\operatorname{up} L, x\in\operatorname{up} X\colon x\in(\operatorname{val} f)_i K\}=\{X\in\mathfrak{A}_i \mid \forall K\in\operatorname{up} L\colon\operatorname{up} X\subseteq(\operatorname{val} f)_i K\}$. [TODO: The same formula as below only with other variable names.] [TODO: Correct order of coords.]

$\operatorname{up}(L\cup\{(i;X)\})\subseteq f\Leftrightarrow\forall K\in\operatorname{up}(L\cup\{(i;X)\})\colon K\in f\Leftrightarrow\forall P\in\operatorname{up} L, X'\in\operatorname{up} X\colon P\cup\{(i;X')\}\in f\Leftrightarrow\forall P\in\operatorname{up} L, X'\in\operatorname{up} X\colon X'\in(\operatorname{val} f)_i P\Leftrightarrow\forall P\in\operatorname{up} L\colon\operatorname{up} X\subseteq(\operatorname{val} f)_i P$

Thus $(\operatorname{val} E^*f)_i L=\{X\in\mathfrak{A}_i \mid \forall P\in\operatorname{up} L\colon\operatorname{up} X\subseteq(\operatorname{val} f)_i P\}=\left\{X\in\mathfrak{A}_i \mid \operatorname{up} X\subseteq\bigcap_{P\in\operatorname{up} L}(\operatorname{val} f)_i P\right\}$. [TODO: First try to prove for the binary case.]

$A\cup B\in(\operatorname{val} E^*f)_i L\Leftrightarrow\left\{X\in\mathfrak{A}_i \mid \operatorname{up}(A\cup B)\subseteq\bigcap_{P\in\operatorname{up} L}(\operatorname{val} f)_i P\right\}\Leftrightarrow\left\{X\in\mathfrak{A}_i \mid \operatorname{up} A\cap\operatorname{up} B\subseteq\bigcap_{P\in\operatorname{up} L}(\operatorname{val} f)_i P\right\}$ (the lemma does not work, try to use a filter base)

$A\cup B\in(\operatorname{val} E^*f)_i L\Leftrightarrow\forall P\in\operatorname{up} L\colon\operatorname{up}(A\cup B)\subseteq(\operatorname{val} f)_i P\Leftrightarrow\forall P\in\operatorname{up} L\colon(\operatorname{up} A\subseteq(\operatorname{val} f)_i P\vee\operatorname{up} B\subseteq(\operatorname{val} f)_i P)\Leftarrow\forall P\in\operatorname{up} L\colon(\operatorname{up} A\subseteq(\operatorname{val} f)_i P\vee\forall P\in\operatorname{up} L\colon\operatorname{up} B\subseteq(\operatorname{val} f)_i P)$ (used the lemma). [TODO: Reverse implication.]

$\forall P\in\operatorname{up} L\colon(\operatorname{up} A\subseteq(\operatorname{val} f)_i P\vee\operatorname{up} B\subseteq(\operatorname{val} f)_i P)\Rightarrow\forall P\in\operatorname{up} L\colon(\operatorname{up} A\cap\operatorname{up} B\subseteq(\operatorname{val} f)_i P)\Leftrightarrow\forall P\in\operatorname{up} L\colon\operatorname{up}(A\cup B)\subseteq(\operatorname{val} f)_i P$.

Thus?? $A\cup B\in(\operatorname{val} E^*f)_i L\Leftrightarrow A\in(\operatorname{val} E^*f)_i L\vee B\in(\operatorname{val} E^*f)_i L$. □

Consider $f\sqcap^{\mathsf{RLD}}\bigcup S=\bigcup\langle f\sqcap^{\mathsf{RLD}}\rangle S$.
1. If f is not required to be complete this formula fails even for set-valued S.
2. Let f is complete. Then it fails for specifically choosen S.

Theorem 36. *If f is a complete reloids and S is a set of complete reloids. Then [TODO: The same for funcoids?]*

$$f\cap^{\mathsf{RLD}}\bigcup S=\bigcup\langle f\cap^{\mathsf{RLD}}\rangle S.$$

Theorem 37. *Composition with a (co?)complete reloid is an adjoint:*

Proof. $F\circ$ is a lower adjoint. Let ξ is its upper adjoint. Then
$F\circ x\subseteq y\Leftrightarrow x\subseteq\xi(y)$
$x\subseteq\xi(F\circ x)\Leftrightarrow F\circ\xi(y)$
$\xi(b)=\max\{x\in\mathsf{RLD}\mid F\circ x\subseteq b\}$ (a proof circle follows this)

$x\subseteq F^{-1}\circ b\Rightarrow F\circ x\subseteq F\circ b$

$F\circ\bigcup^{\mathsf{RLD}}R=\bigcap^{\mathsf{RLD}}\{F\circ K\mid K\in\operatorname{up}\bigcup^{\mathsf{RLD}}R\}$
$\bigcup^{\mathsf{RLD}}\langle F\circ\rangle R=\bigcup^{\mathsf{RLD}}\{\bigcap^{\mathsf{RLD}}\{F\circ G\mid G\in\operatorname{up}g\}\mid g\in R\}$ □

Conjecture 38. $\operatorname{Compl}f\cap^{\mathsf{RLD}}\operatorname{Compl}g=\operatorname{Compl}(f\cap^{\mathsf{RLD}}g)$ for every reloids f and g.

Proof. $\operatorname{Compl}(f\cap^{\mathsf{RLD}}g)=\bigcup^{\mathsf{RLD}}\{(f\cap^{\mathsf{RLD}}g)|^{\mathsf{RLD}}_{\{\alpha\}}\mid\alpha\in\mho\}$
$\operatorname{Compl}f\cap^{\mathsf{RLD}}\operatorname{Compl}g=\bigcup^{\mathsf{RLD}}\{f|^{\mathsf{RLD}}_{\{\alpha\}}\mid\alpha\in\mho\}\cap^{\mathsf{RLD}}\bigcup^{\mathsf{RLD}}\{g|^{\mathsf{RLD}}_{\{\alpha\}}\mid\alpha\in\mho\}$
$(\operatorname{Compl}(f\cap^{\mathsf{RLD}}g))|^{\mathsf{RLD}}_{\{\beta\}}=(f\cap^{\mathsf{RLD}}g)|^{\mathsf{RLD}}_{\{\beta\}}$
$(\operatorname{Compl}f\cap^{\mathsf{RLD}}\operatorname{Compl}g)|^{\mathsf{RLD}}_{\{\beta\}}=(f\cap^{\mathsf{RLD}}g)|^{\mathsf{RLD}}_{\{\beta\}}$.
So enough to prove that $\operatorname{Compl}f\cap^{\mathsf{RLD}}\operatorname{Compl}g$ is complete.
Let $A=\operatorname{atoms}^{\mathsf{ComplRLD}}f$ and $B=\operatorname{atoms}^{\mathsf{ComplRLD}}g$. Then ??
Obviously $\operatorname{Compl}f\cap^{\mathsf{RLD}}\operatorname{Compl}g\supseteq\operatorname{Compl}f\cap^{\mathsf{ComplRLD}}\operatorname{Compl}g$
Suppose it exists $a\in\operatorname{atoms}^{\mathsf{RLD}}(\operatorname{Compl}f\cap^{\mathsf{RLD}}\operatorname{Compl}g)$ such that $a\notin$ toms$^{\mathsf{RLD}}(\operatorname{Compl}f\cap^{\mathsf{ComplRLD}}\operatorname{Compl}g)$. Then ?? □

Conjecture 39. *If f and g are reloids, then*

$$g\circ f=\bigcup^{\mathsf{RLD}}\{G\circ F\mid F\in\operatorname{atoms}^{\mathsf{RLD}}f,G\in\operatorname{atoms}^{\mathsf{RLD}}g\}.$$

Proof. $g\circ f=??=g\circ\bigcup^{\mathsf{RLD}}\operatorname{atoms}^{\mathsf{RLD}}f=??=\bigcup^{\mathsf{RLD}}\langle g\circ\rangle\operatorname{atoms}^{\mathsf{RLD}}f$

$\bigcup^{\mathsf{RLD}}\{G\circ F\mid F\in\operatorname{atoms}^{\mathsf{RLD}}f,G\in\operatorname{atoms}^{\mathsf{RLD}}g\}s\subseteq\bigcup^{\mathsf{RLD}}\{g\circ F\mid F\in\operatorname{atoms}^{\mathsf{RLD}}f\}\subseteq g\circ f$
?? □

Theorem 40. $\operatorname{dom}(\mathsf{RLD})_{\mathrm{in}}f=\operatorname{dom}f$ *and* $\operatorname{im}(\mathsf{RLD})_{\mathrm{in}}f=\operatorname{im}f$ *for every funcoid f.*

Proof. Let an atomic f.o. $a\subseteq\operatorname{dom}f$. Then exists atomic f.o. $b\subseteq\operatorname{im}f$ such that $a\times^{\mathsf{FCD}}b\subseteq f$. Consequently
$a\times^{\mathsf{RLD}}b\subseteq(\mathsf{RLD})_{\mathrm{in}}f\Rightarrow\forall K\in\operatorname{up}(\mathsf{RLD})_{\mathrm{in}}f:a\times^{\mathsf{RLD}}b\subseteq K\Rightarrow\forall K\in\operatorname{up}(\mathsf{RLD})_{\mathrm{in}}f:a\subseteq\operatorname{dom}K\Leftrightarrow a\subseteq\bigcap^{\mathfrak{F}}\langle\operatorname{dom}\rangle\operatorname{up}(\mathsf{RLD})_{\mathrm{in}}f\Leftrightarrow a\subseteq\operatorname{dom}(\mathsf{RLD})_{\mathrm{in}}f$.
Let now an atomic f.o. $a\subseteq\operatorname{dom}(\mathsf{RLD})_{\mathrm{in}}f$. Then $\forall K\in\operatorname{up}(\mathsf{RLD})_{\mathrm{in}}f:a\subseteq\operatorname{dom}K$
What is equivalent to

$$\forall K\in\bigcap\{\operatorname{up}(a\times^{\mathsf{RLD}}b)\mid a,b\in\operatorname{atoms}^{\mathfrak{F}}\mho,a\times^{\mathsf{FCD}}b\subseteq f\}:a\subseteq\operatorname{dom}K$$

Let $K\in\operatorname{up}f$. Then $K\supseteq a\times^{\mathsf{FCD}}b$ for every $a,b\in\operatorname{atoms}^{\mathfrak{F}}\mho$ where $a\times^{\mathsf{FCD}}b\subseteq f$ that is exist ?? $K\in\operatorname{up}(a\times^{\mathsf{RLD}}b)$ for ??
??
from what follows?? [FIXME: b is depended on K] that exist $b\subseteq\operatorname{im}f$ such that $\forall K\in\operatorname{up}(\mathsf{RLD})_{\mathrm{in}}f:a\times^{\mathsf{RLD}}b\subseteq K$ that is $\forall K\in\operatorname{up}(\mathsf{RLD})_{\mathrm{in}}f:K\in\operatorname{up}(a\times^{\mathsf{RLD}}b)$ and thus $(\mathsf{RLD})_{\mathrm{in}}f\supseteq a\times^{\mathsf{RLD}}b$ and consequently $\operatorname{dom}(\mathsf{RLD})_{\mathrm{in}}f\supseteq\operatorname{dom}(a\times^{\mathsf{RLD}}b)=a$.

Thus $a\subseteq\operatorname{dom}f\Leftrightarrow a\subseteq\operatorname{dom}(\mathsf{RLD})_{\mathrm{in}}f$ for each atomic f.o. a from what follows $\operatorname{dom}(\mathsf{RLD})_{\mathrm{in}}f=\operatorname{dom}f$.

$\operatorname{im}(\mathsf{RLD})_{\mathrm{in}} f = \operatorname{im} f$ is similar. □

Theorem 41. *A reloid f is monovalued iff $\forall g \in \mathsf{RLD}: (g \subseteq f \Rightarrow g = f|^{\mathsf{RLD}}_{\operatorname{dom} g})$.*

Proof.

⇒. Let f is monovalued. Then exists $F \in \operatorname{up} f$ such that $f = F|^{\mathsf{RLD}}_{\operatorname{dom} f}$. Let $g \in \mathsf{RLD}$ and $g \subseteq f$. Then exists $G \in \operatorname{up} g$ such that $g = G|^{\mathsf{RLD}}_{\operatorname{dom} g}$. We have $g = g \cap^{\mathsf{RLD}} f = G|^{\mathsf{RLD}}_{\operatorname{dom} g} \cap^{\mathsf{RLD}} F|^{\mathsf{RLD}}_{\operatorname{dom} f} = G|^{\mathsf{RLD}}_{\operatorname{dom} g} \cap^{\mathsf{RLD}} F|^{\mathsf{RLD}}_{\operatorname{dom} g} = (G \cap^{\mathsf{RLD}} F)|^{\mathsf{RLD}}_{\operatorname{dom} g} \supseteq f|^{\mathsf{RLD}}_{\operatorname{dom} g}$. But obviously $g \subseteq f|^{\mathsf{RLD}}_{\operatorname{dom} g}$. So $g = f|^{\mathsf{RLD}}_{\operatorname{dom} g}$. □

Conjecture 42. $\operatorname{Compl} f = f \setminus^{*\mathsf{FCD}} (\Omega \times^{\mathsf{FCD}} \mho)$ *for every funcoid f.*

This conjecture may be proved by considerations similar to these in the section "Fréchet filter" in [?].

Example 43. $(\mathsf{RLD})_{\mathrm{in}}$ is not a lower adjoint (in general).

Proof. Enough to prove one of the following:

Enough to prove non-existence of $\max\{f \in \mathsf{FCD} \mid (\mathsf{RLD})_{\mathrm{in}} f \subseteq g\}$ for some reloid g.

??Enough to prove $(\mathsf{RLD})_{\mathrm{in}} \bigcup^{\mathsf{FCD}} S \neq \bigcup^{\mathsf{FCD}} \langle (\mathsf{RLD})_{\mathrm{in}} \rangle S$ for some set S of funcoids.

?? □

Theorem 44. *A filter $\mathcal{A}$ is connected regarding a reloid f iff it is connected regarding the funcoid $(\mathsf{FCD}) f$.*

Proof. $\mathcal{A}$ is connected regarding f iff $\mathcal{A}$ is connected regarding every element of $F \in \operatorname{up} f$ (considered as reloids) that is iff $S^*(F \cap^{\mathsf{RLD}} (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{A})) = \mathcal{A} \times^{\mathsf{RLD}} \mathcal{A}$.

?? □

Theorem 45. $(\mathsf{RLD})_{\mathrm{out}} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \neq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ *in general.*

Proof. $(\mathsf{RLD})_{\mathrm{out}} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = \bigcap^{\mathsf{RLD}} \operatorname{up}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})$

If the equality holds, $\forall F \in \operatorname{up}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \exists A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}: A \times B \subseteq F$.

Let $\mathcal{L}$, $\mathcal{B}$ are f.o. and the cardinality of the set $H = \{\mathcal{Y} \in \mathfrak{F} \mid \mathcal{B} \subset \mathcal{Y} \subseteq \mathcal{L}\}$ is infinite but not greater than cardinality of a set A. (We can always assume existence of such set A, extending the base set $\mho$ above if necessary.)

For every f.o. $\mathcal{Y} \in H$ choose a set $Y_{\mathcal{Y}} \in \operatorname{up} \mathcal{B}$ such that $Y_{\mathcal{Y}} \notin \operatorname{up} \mathcal{Y}$

Let z is a surjection from A to H.

Consider the binary relation $F = \bigcup \{\{\alpha\} \times Y_{z\alpha} \mid \alpha \in A\}$.

We have $F \supseteq A \times^{\mathsf{FCD}} \mathcal{B}$ that is $F \in \operatorname{up}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})$.

But if $B \in \operatorname{up} \mathcal{B}$ then ?? $A \times B \nsubseteq F$ because $F\alpha = Y_{z\alpha} \notin \operatorname{up} z\alpha$ that is $\forall \mathcal{Y} \supset \mathcal{B}: Y_{z\alpha} \notin \operatorname{up} \mathcal{Y}$

[Not needed] Consider funcoid $f = \bigcup^{\mathsf{FCD}} \{\{\alpha\} \times^{\mathsf{FCD}} z\alpha \mid \alpha \in A\}$.

?? □

Theorem 46. $\bigcap^{Z(D\mathcal{A})} \{X \cap^{\mathfrak{F}} \mathcal{A} \mid X \in T\} = \mathcal{A} \cap^{\mathfrak{F}} \bigcap T$.

Proof. That $\mathcal{A} \cap^{\mathfrak{F}} \bigcap T$ is a lower bound of $\{X \cap^{\mathfrak{F}} \mathcal{A} \mid X \in T\}$ is obvious.

We need to prove that it is the greatest lower bound, that is that for every lower bound $\mathcal{B} \in Z(D\mathcal{A})$ of $\{X \cap^{\mathfrak{F}} \mathcal{A} \mid X \in T\}$ we have $\mathcal{A} \cap^{\mathfrak{F}} \bigcap T \supseteq \mathcal{B}$.

Let $\mathcal{B} = B \cap^{\mathfrak{F}} \mathcal{A}$ is a lower bound of $\{X \cap^{\mathfrak{F}} \mathcal{A} \mid X \in T\}$ that is $\forall X \in T: B \cap^{\mathfrak{F}} \mathcal{A} \subseteq X \cap^{\mathfrak{F}} \mathcal{A}$. Left to prove that $\mathcal{A} \cap^{\mathfrak{F}} \bigcap T \supseteq B \cap^{\mathfrak{F}} \mathcal{A}$.

$$(X \cup B) \cap^{\mathfrak{F}} \mathcal{A} = (X \cap^{\mathfrak{F}} \mathcal{A}) \cup^{\mathfrak{F}} (B \cap^{\mathfrak{F}} \mathcal{A}) = X \cap^{\mathfrak{F}} \mathcal{A}$$

$$B \cup^{\mathfrak{F}} (\mathcal{A} \cap^{\mathfrak{F}} \bigcap T) = (B \cup^{\mathfrak{F}} \mathcal{A}) \cap^{\mathfrak{F}} (B \cup \bigcap T)$$

$\bigcap^{Z(D\mathcal{A})} \{X \cap^{\mathfrak{F}} \mathcal{A} \mid X \in T\} = \bigcap^{Z(D\mathcal{A})} \{(X \cup B) \cap^{\mathfrak{F}} \mathcal{A} \mid X \in T\} \supseteq B \cap^{\mathfrak{F}} \mathcal{A}.$

Suppose that $\mathcal{A} \cap^{\mathfrak{F}} \bigcap T \not\supseteq B \cap^{\mathfrak{F}} \mathcal{A}$. Then exists $K \in \operatorname{up}(\mathcal{A} \cap^{\mathfrak{F}} \bigcap T)$ such that $K \notin \operatorname{up}(B \cap^{\mathfrak{F}} \mathcal{A})$. That is $\exists L \in \operatorname{up} \mathcal{A}: K = L \cap \bigcap T$ and $\nexists L \in \operatorname{up} \mathcal{A}: K = L \cap B$.

For every $X \in T$ we have $\operatorname{up}(B \cap^{\mathfrak{F}} \mathcal{A}) \supseteq \operatorname{up}(X \cap^{\mathfrak{F}} \mathcal{A})$ that is $\forall Y \in \operatorname{up}(X \cap^{\mathfrak{F}} \mathcal{A}): Y \in \operatorname{up}(B \cap^{\mathfrak{F}} \mathcal{A})$; $\forall P \in \operatorname{up} \mathcal{A}: P \cap X \in \operatorname{up}(B \cap^{\mathfrak{F}} \mathcal{A})$; $\forall P \in \operatorname{up} \mathcal{A} \exists Q \in \operatorname{up} \mathcal{A}: P \cap X = B \cap Q$.

Thus $\exists Q \in \operatorname{up} \mathcal{A}: L \cap X = B \cap Q$

$(B \cap^{\mathfrak{F}} X) \cap^{\mathfrak{F}} \mathcal{A} = (B \cap^{\mathfrak{F}} \mathcal{A}) \cap^{\mathfrak{F}} X = B \cap^{\mathfrak{F}} \mathcal{A}.$
$\forall X \in T: (B \cap^{\mathfrak{F}} X) \cap^{\mathfrak{F}} \mathcal{A} \subseteq X \cap^{\mathfrak{F}} \mathcal{A}$
??

$(B \cap^{\mathfrak{F}} \mathcal{A}) \cap^{\mathfrak{F}} X = (B \cap^{\mathfrak{F}} \mathcal{A}) \cap^{\mathfrak{F}} (X \cap^{\mathfrak{F}} \mathcal{A}) = B \cap^{\mathfrak{F}} \mathcal{A}$
?? □

Theorem 47. $\bigcup^{Z(D\mathcal{A})} \{X \cap^{\mathfrak{F}} \mathcal{A} \mid X \in T\} = \mathcal{A} \cap^{\mathfrak{F}} \bigcup T.$

Proof. ?? □

Lemma 48. *If staroid $0 \neq f \sqsubseteq a^n_{\mathsf{Strd}}$ for an ultrafilter a and an index set n, then $n \times \{a\} \in \operatorname{GR} f$. [TODO: Can be generalized for arbitrary staroidal products?]*

Proof. If $K_i \not\sqsupseteq a_i$ for some $i \in n$ then $K \notin \operatorname{GR} a^n_{\mathsf{Strd}}$ and thus $K \notin \operatorname{GR} f$.

Suppose that for every K such that $K_i \sqsupseteq a_i$ for every $i \in n$ we have $K \notin \operatorname{GR} f$. Then $\operatorname{GR} f = \emptyset$ what is impossible.

Thus there exists K such that $K_i \sqsupseteq a_i$ for every $i \in n$ and $K \in \operatorname{GR} f$.

By the previous lemma, $\lambda i \in n: K_i \sqcap a \in \operatorname{GR} f$ that is $n \times \{a\} \in \operatorname{GR} f$. □

Theorem 49. $\operatorname{id}^{\mathsf{Strd}}_{a[n]}$ *is an atomic ?? if a is an atomic filter.*

Proof. Suppose $0 \neq f \sqsubseteq \operatorname{id}^{\mathsf{Strd}}_{a[n]}$. Then $f \sqsubseteq a^n_{\mathsf{Strd}}$ and thus by the lemma $n \times \{a\} \in \operatorname{GR} f$.

We need to prove that $f \sqsupseteq \operatorname{id}^{\mathsf{Strd}}_{a[n]}$ that is $L \in \operatorname{GR} f \Leftarrow L \in \operatorname{id}^{\mathsf{Strd}}_{a[n]}$ that is $L \in \operatorname{GR} f \Leftarrow \bigsqcap^3_{i \in n} L_i \in \partial a$.

Really, $\bigsqcap^3_{i \in n} L_i \in \partial a \Rightarrow \forall i \in n: L_i \in \partial a \Rightarrow \forall i \in n: L_i \sqsupseteq a \Rightarrow L \in \operatorname{GR} f$. □

7 Misc

Lemma 50. *$x \times^{\mathsf{FCD}} y \subseteq F \wedge x \times^{\mathsf{FCD}} y \subseteq G \Rightarrow x \times^{\mathsf{FCD}} y \subseteq F \cap G$ for any atomic filters x and y and binary relations F and G.*

Proof. $x \times^{\mathsf{FCD}} y \subseteq F \Leftrightarrow \langle F \rangle x \supseteq y \Leftrightarrow \forall X \in \operatorname{up} x: \langle F \rangle X \supseteq y \Leftrightarrow \forall X \in \operatorname{up} x: X \not\asymp \langle F^{-1} \rangle y$. □

Consider an infinite lineraly ordered set.
Let a is a nontrivial atomic filter. Choose atomic reloid f in $(a \times a) \cap (>)$.
$f \circ f^{-1} \subseteq ((a \times a) \cap (>)) \circ ((a \times a) \cap (<)) \subseteq a \times a$
$y > x$, $z < y$
??$f \circ f^{-1} \subseteq 1_{\operatorname{Dst} f}$

A space is compact if and only if every collection of closed sets satisfying the finite intersection property has nonempty intersection itself. (See here).

Theorem 51. *If $S \in \mathscr{P}\mathfrak{F}^2$ then*

$$\bigcap^{\mathfrak{F}} \{\mathcal{A} \times \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S\} = \bigcap^{\mathfrak{F}} \operatorname{dom} S \times \bigcap^{\mathfrak{F}} \operatorname{im} S.$$

Proof. Let $\mathcal{P} \in \operatorname{dom} S$, $\mathcal{Q} \in \operatorname{im} S$. Then there exist $\mathcal{P}'$ and $\mathcal{Q}'$ such that $(\mathcal{P}, \mathcal{Q}') \in S$, $(\mathcal{P}', \mathcal{Q}) \in S$. $\mathcal{P} \times \mathcal{Q}' \cap \mathcal{P}' \times \mathcal{Q} = (\mathcal{P} \cap \mathcal{P}') \times (\mathcal{Q} \cap \mathcal{Q}')$ (used the previous theorem). $\bigcap^{\mathfrak{F}}\{\mathcal{A} \times \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S\} \subseteq \mathcal{P} \times \mathcal{Q}' \cap \mathcal{P}' \times \mathcal{Q} = (\mathcal{P} \cap \mathcal{P}') \times (\mathcal{Q} \cap \mathcal{Q}') \subseteq \mathcal{P} \times \mathcal{Q}$. This implies that

$$\begin{aligned}\bigcap^{\mathfrak{F}}\{\mathcal{A} \times \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S\} &= \bigcap^{\mathfrak{F}}\{\mathcal{A} \times \mathcal{B} \mid \mathcal{A} \in \operatorname{dom} S, \mathcal{B} \in \operatorname{im} S\} \\ &= \bigcap^{\mathfrak{F}}\{A \times B \mid A \in \langle \mathrm{up} \rangle \operatorname{dom} S, B \in \langle \mathrm{up} \rangle \operatorname{im} S\}.\end{aligned}$$

On the other side

$\bigcap^{\mathfrak{F}} \operatorname{dom} S \times \bigcap^{\mathfrak{F}} \operatorname{im} S = \bigcap^{\mathfrak{F}}\{A \times B \mid A \in \mathrm{up} \bigcap^{\mathfrak{F}} \operatorname{dom} S, B \in \mathrm{up} \bigcap^{\mathfrak{F}} \operatorname{im} S\}$

??

If $A \in \langle \mathrm{up} \rangle \operatorname{dom} S$ then $A \in \mathrm{up} \bigcap^{\mathfrak{F}} \operatorname{dom} S$. If $A \in \mathrm{up} \bigcap^{\mathfrak{F}} \operatorname{dom} S$ then □

$\forall S \in \mathscr{P}\mathfrak{F}: (\mathcal{F} \cap \bigcup^{\mathfrak{F}} S \neq \emptyset \Rightarrow \exists \mathcal{K} \in S: \mathcal{F} \cap \mathcal{K} \neq \emptyset)$

$\forall S \in \mathscr{P}\mathfrak{F}: (\langle f \rangle A \cap \bigcup^{\mathfrak{F}} S \neq \emptyset \Rightarrow \exists \mathcal{K} \in S: \langle f \rangle A \cap \mathcal{K} \neq \emptyset)$

$\forall S \in \mathscr{P}\mathfrak{F}: (A\, [f] \bigcup^{\mathfrak{F}} S \Rightarrow \exists \mathcal{K} \in S: A\, [f]\, \mathcal{K})$

$\forall S \in \mathscr{P}\mathfrak{F}: \langle f \rangle \bigcup^{\mathfrak{F}} S = \bigcup^{\mathfrak{F}} \langle \langle f \rangle \rangle S$;

$\forall S \in \mathscr{P}\mathscr{P}\mho: \langle f \rangle \bigcup S = \bigcup^{\mathfrak{F}} \langle \langle f \rangle \rangle S$

Proposition 52. *Equivalence of morphisms is an equivalence relation.*

Proof.

Reflexity. Follows from the identity.

Symmetry. Obvious.

Transitivity. Let $f \sim g$ and $g \sim h$. Then there exist a morphism p such that $\operatorname{Src} p \sqsubseteq \operatorname{Src} f$, $\operatorname{Src} p \sqsubseteq \operatorname{Src} g$, $\operatorname{Dst} p \sqsubseteq \operatorname{Dst} f$, $\operatorname{Dst} p \sqsubseteq \operatorname{Dst} g$ and $\iota_{\operatorname{Src} f, \operatorname{Dst} f}\, p = f$ and $\iota_{\operatorname{Src} g, \operatorname{Dst} g}\, p = g$ and ??

??

$f = \iota_{\operatorname{Src} f, \operatorname{Dst} f}\, p = (\operatorname{Dst} p \hookrightarrow \operatorname{Dst} f) \circ p \circ (\operatorname{Src} p \hookrightarrow \operatorname{Src} f)^{\dagger}$

$g = \iota_{\operatorname{Src} g, \operatorname{Dst} g}\, p = (\operatorname{Dst} p \hookrightarrow \operatorname{Dst} g) \circ p \circ (\operatorname{Src} p \hookrightarrow \operatorname{Src} g)^{\dagger}$

$g = \iota_{\operatorname{Src} g, \operatorname{Dst} g}\, q = (\operatorname{Dst} q \hookrightarrow \operatorname{Dst} g) \circ q \circ (\operatorname{Src} q \hookrightarrow \operatorname{Src} g)^{\dagger}$

$h = \iota_{\operatorname{Src} h, \operatorname{Dst} h}\, q = (\operatorname{Dst} q \hookrightarrow \operatorname{Dst} h) \circ q \circ (\operatorname{Src} q \hookrightarrow \operatorname{Src} h)^{\dagger}$

??

We have like:

$(X \hookrightarrow A) \circ p = (Y \hookrightarrow A) \circ q$ and need $z \sqsubseteq p, q$ such that

$$(\operatorname{Dst} z \hookrightarrow A) \circ z = (X \hookrightarrow A) \circ p = (Y \hookrightarrow A) \circ q.$$

Take $z = p \sqcap q$. Repeating this, we get:

$g = \iota_{\operatorname{Src} g, \operatorname{Dst} g}\, p = (\operatorname{Dst} z \hookrightarrow \operatorname{Dst} g) \circ z \circ (\operatorname{Src} z \hookrightarrow \operatorname{Src} g)^{\dagger}$

$g = \iota_{\operatorname{Src} g, \operatorname{Dst} g}\, q = (\operatorname{Dst} z \hookrightarrow \operatorname{Dst} g) \circ z \circ (\operatorname{Src} z \hookrightarrow \operatorname{Src} g)^{\dagger}$

??

Axiom: $X \hookrightarrow A$ is metamonovalued (requires order on the set of **all** objects). □

Conjecture 53. $(\mathsf{RLD})_{\mathrm{in}} f = \bigsqcap^{\mathsf{RLD}} \mathrm{up}^{\Gamma(\operatorname{Src} f; \operatorname{Dst} f)} f$ *for every funcoid* f.

Proof. Let $K \in (\mathsf{RLD})_{\mathrm{in}} f$.

$\bigsqcap^{\mathsf{RLD}} \mathrm{up}^{\Gamma(\operatorname{Src} f; \operatorname{Dst} f)} f = \bigsqcap^{\mathsf{RLD}} \mathrm{up}^{\Gamma(\operatorname{Src} f; \operatorname{Dst} f)} \bigsqcup \operatorname{atoms} f \sqsubseteq \bigsqcap^{\mathsf{RLD}} \mathrm{up}^{\Gamma(\operatorname{Src} f; \operatorname{Dst} f)} \bigsqcup^{\mathsf{FCD}} \{X_{\mathcal{A}} \times Y_{\mathcal{B}} \mid \mathcal{A} \in \mathfrak{F}(\operatorname{Src} f), \mathcal{B} \in \mathfrak{F}(\operatorname{Dst} f), \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \sqsubseteq f\} = \bigsqcap^{\mathsf{RLD}} \mathrm{up}^{\Gamma(\operatorname{Src} f; \operatorname{Dst} f)} \bigcup \{X_{\mathcal{A}} \times Y_{\mathcal{B}} \mid \mathcal{A} \in \mathfrak{F}(\operatorname{Src} f), \mathcal{B} \in \mathfrak{F}(\operatorname{Dst} f), \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \sqsubseteq f\}$

Pattern theorem 8.19. □

Conjecture 54. $(\mathsf{FCD}) f = \bigsqcap^{\mathsf{FCD}} (\Gamma(A; B) \cap \operatorname{GR} f)$ *for every reloid* $f \in \mathsf{RLD}(A; B)$.

Proof. $x \times^{\mathsf{FCD}} y \in \operatorname{atoms} (\mathsf{FCD}) f \Leftrightarrow x \times^{\mathsf{RLD}} y \not\asymp f \Leftrightarrow$??

$x \times^{\mathsf{FCD}} y \in \operatorname{atoms} \prod^{\mathsf{FCD}} (\Gamma(A; B) \cap \operatorname{GR} f) \Leftrightarrow \forall K \in \Gamma(A; B) \cap \operatorname{GR} f: x \times^{\mathsf{FCD}} y \in \operatorname{atoms} K \Leftarrow \forall K \in \operatorname{GR} f: x \times^{\mathsf{FCD}} y \in \operatorname{atoms} K \Leftrightarrow x \times^{\mathsf{RLD}} y \not\asymp f$.

??

It's enough to prove $\operatorname{up}^{\Gamma(A;B)} (\mathsf{FCD})\, f = \Gamma(A; B) \cap \operatorname{GR} f$.

Really, $\operatorname{up}^{\Gamma(A;B)} (\mathsf{FCD})\, f = \operatorname{up}^{\Gamma(A;B)} \prod^{\mathsf{FCD}} \operatorname{GR} f$

$K \in \operatorname{up}^{\Gamma(A;B)} \prod^{\mathsf{FCD}} \operatorname{GR} f \Rightarrow K \sqsupseteq \prod^{\mathsf{FCD}} \operatorname{GR} f$ [TODO: To continue this implication we MUST use that $K \in \Gamma$ (otherwise take thick identity as a counterexample)]

?? □

Theorem 55. *$b \not\asymp^{\operatorname{Anch}(\mathfrak{A})} \operatorname{StarComp}(a; f) \Leftrightarrow \forall A \in \operatorname{GR} a, B \in \operatorname{GR} b, i \in n: A_i\, [f_i]\, B_i$ for anchored relations a and b, provided that $\operatorname{Src} f_i$ are atomic posets.*

Proof.

$$
\begin{aligned}
&b \not\asymp^{\operatorname{Anch}(\mathfrak{A})} \operatorname{StarComp}(a; f) \Leftrightarrow\\
&\exists x \in \operatorname{Anch}(\mathfrak{A}) \setminus \{\bot\}: (x \sqsubseteq b \wedge x \sqsubseteq \operatorname{StarComp}(a; f)) \Leftrightarrow\\
&\exists x \in \operatorname{Anch}(\mathfrak{A}) \setminus \{\bot\}: (x \sqsubseteq b \wedge \forall B \in \operatorname{GR} x: B \in \operatorname{GR} \operatorname{StarComp}(a; f)) \Leftrightarrow\\
&\exists x \in \operatorname{Anch}(\mathfrak{A}) \setminus \{\bot\}: (x \sqsubseteq b \wedge \forall B \in \operatorname{GR} x: (\lambda i \in n: \langle {f_i}^{-1} \rangle B_i) \in \operatorname{GR} a) \Leftrightarrow \text{ remove?}\\
&\exists x \in \operatorname{Anch}(\mathfrak{A}) \setminus \{\bot\}: (\forall B \in \operatorname{GR} x: B \in \operatorname{GR} b \wedge \forall B \in \operatorname{GR} x: (\lambda i \in n: \langle {f_i}^{-1} \rangle B_i) \in \operatorname{GR} a) \Leftrightarrow\\
&\exists x \in \operatorname{Anch}(\mathfrak{A}) \setminus \{\bot\}\, \forall B \in \operatorname{GR} x: (B \in \operatorname{GR} b \wedge (\lambda i \in n: \langle {f_i}^{-1} \rangle B_i) \in \operatorname{GR} a) \Leftrightarrow\\
&??\\
&\forall A \in \operatorname{GR} a, B \in \operatorname{GR} b, i \in n: A_i \not\asymp \langle {f_i}^{-1} \rangle B_i \Leftrightarrow\\
&\forall A \in \operatorname{GR} a, B \in \operatorname{GR} b, i \in n: A_i\, [f_i]\, B_i. \Leftrightarrow\\
&??\\
&\exists x \in \operatorname{Anch}(\mathfrak{A}) \setminus \{\bot\}: (\forall A \in \operatorname{GR} x: (A \in \operatorname{GR} a \wedge (\lambda i \in n: \langle f_i \rangle A_i) \in \operatorname{GR} b))\\
&\exists x \in \operatorname{Anch}(\mathfrak{A}) \setminus \{\bot\}: (x \sqsubseteq a \wedge \forall A \in \operatorname{GR} x: (\lambda i \in n: \langle f_i \rangle A_i) \in \operatorname{GR} b)\\
&\exists x \in \operatorname{Anch}(\mathfrak{A}) \setminus \{\bot\}: (x \sqsubseteq a \wedge \forall A \in \operatorname{GR} x: A \in \operatorname{GR} \operatorname{StarComp}(b; f^{\dagger}))\\
&\exists x \in \operatorname{Anch}(\mathfrak{A}) \setminus \{\bot\}: (x \sqsubseteq a \wedge x \sqsubseteq \operatorname{StarComp}(b; f^{\dagger})) \Leftrightarrow\\
&a \not\asymp^{\operatorname{Anch}(\mathfrak{A})} \operatorname{StarComp}(b; f^{\dagger})
\end{aligned}
$$

□