Categories related with funcoids

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July 26, 2012

Abstract

I consider some categories related with pointfree funcoids.

1 Draft status

This is a rough partial draft.

2 Topic of this article

In this article are considered some categories related to *pointfree funcoids* [1].

3 Category of continuous morphisms

I will denote $\operatorname{Ob} f$ the object (source and destination) of an endomorphism f.

Definition 1. Let C is a partially ordered category. The category $\mathbf{cont}(C)$ (which I call *the category of continuous morphism* over C) is:

- Objects are endomorphisms of category C.
- Morphisms are triples $(f; a; b)$ where a and b are objects and $f: \operatorname{Ob} a \to \operatorname{Ob} b$ is a morphism of the category C such that $f \circ a \sqsubseteq b \circ f$.
- Composition of morphisms is defined by the formula $(g; b; c) \circ (f; a; b) = (g \circ f; a; c)$.
- Identity morphisms are $(a; a; 1_a^C)$.

It is really a category:

Proof. We need to prove that: composition of morphisms is a morphism, composition is associative, and identity morphisms can be canceled on the left and on the right.

That composition of morphisms is a morphism follows from these implications:

$$f \circ a \sqsubseteq b \circ f \wedge g \circ b \sqsubseteq c \circ g \Rightarrow g \circ f \circ a \sqsubseteq g \circ b \circ f \sqsubseteq c \circ g \circ f.$$

That composition is associative is obvious.

That identity morphisms can be canceled on the left and on the right is obvious. □

Remark 2. The "physical" meaning of this category is:

- Objects (endomorphisms of C) are spaces.
- Morphisms are continuous functions between spaces.
- $f \circ a \sqsubseteq b \circ f$ intuitively means that f combined with an infinitely small is less than infinitely small combined with f (that is f is continuous).

Remark 3. Every $\operatorname{Hom}(\mathfrak{A}; \mathfrak{B})$ of **Pos** is partially ordered by the formula $a \leqslant b \Leftrightarrow \forall x \in \mathfrak{A}: a(x) \leqslant b(x)$. So $\mathbf{cont}(\mathbf{Pos})$ is defined.

Definition 4. I call a **Pos**-morphism *monovalued* when it maps atoms to atoms or least element.

Definition 5. I call a **Pos**-morphism *entirely defined* when its value is non-least on every non-least element.

Obvious 6. A morphism is both monovalued and entirely defined iff it maps atoms into atoms.

[TODO: Show how it relates with dagger categories.]

Definition 7. **mePos** is the subcategory of **Pos** with only monovalued and entirely defined morphisms.

Obvious 8. This is a well defined category.

Definition 9. **mefp**FCD is the subcategory of **fp**FCD with only monovalued and entirely defined morphisms.

Remark 10. In the two above definitions different definitions of monovaluedness and entire definedness from different articles.

4 Definition of the categories

Definition 11. A *(pointfree) endo-funcoid* is a (pointfree) funcoid with the same source and destination (an endomorphism of the category of (pointfree) funcoids). I will denote $\operatorname{Ob} f$ the object of an endomorphism f.

Obvious 12. The *category of continuous pointfree funcoids* **cont**(**fpFCD**) is:

- Objects are small pointfree endo-funcoids.
- Morphisms from an object a to an object b are triples $(f; a; b)$ where f is a pointfree funcoid from $\operatorname{Ob} a$ to $\operatorname{Ob} b$ such that f is a continuous morphism from a to b (that is $f \circ a \sqsubseteq b \circ f$, or equivalently $a \sqsubseteq f^{-1} \circ b \circ f$, or equivalently $f \circ a \circ f^{-1} \sqsubseteq f$).
- Composition is the composition of pointfree funcoids.
- Identity for an object a is $(I^{\mathsf{FCD}}_{\operatorname{Ob} a}; a; a)$.

5 Isomorphisms

Theorem 13. If f is an isomorphism $a \rightarrow b$ of the category **cont**(**fpFCD**), then:

1. $f \circ a = b \circ f$;
2. $a = f^{-1} \circ b \circ f$;
3. $f \circ a \circ f^{-1} = b$.

Proof. Note that f is monovalued and entirely defined.

1. We have $f \circ a \sqsubseteq b \circ f$ and $f^{-1} \circ b \sqsubseteq a \circ f^{-1}$. Consequently $f^{-1} \circ f \circ a \sqsubseteq f^{-1} \circ b \circ f$; $a \sqsubseteq f^{-1} \circ b \circ f$; $a \circ f^{-1} \sqsubseteq f^{-1} \circ b \circ f \circ f^{-1}$; $a \circ f^{-1} \sqsubseteq f^{-1} \circ b$. Similarly $b \circ f \sqsubseteq f \circ a$. So $f \circ a = b \circ f$.

2 and 3. Follow from the definition of isomorphism. □

Isomorphisms are meant to preserve structure of objects. I will show that (under certain conditions) isomorphisms of **cont**(**fpFCD**) really preserve structure of objects.

First we will consider an isomorphism between objects a and b which are funcoids (not the general case of pointfree funcoids). In this case a map which preserves structure of objects is a *bijection*. It is really a bijection as the following theorem says:

Theorem 14. If f is an isomorphism of the category of funcoids then f is a discrete funcoid (so, it is essentially a bijection). [TODO: Split it into two propositions: about completeness and co-completeness.]

Proof. $\langle f\rangle^* A \sqcap \langle f\rangle^*((\mathrm{Src}\, f) \setminus A) = 0^{\mathrm{Dst}\, f}$ because f is monovalued.

$\langle f\rangle^* A \sqcup \langle f\rangle^*((\mathrm{Src}\, f) \setminus A) = 1^{\mathrm{Dst}\, f}$.

Therefore $\langle f\rangle^* A$ is a principal filter (theorem 49 in [2]). So f is co-complete.

That f is complete follows from symmetry. □

For wider class of pointfree funcoids the concept of bijection does not make sense. Instead we would want a structure preserving map to be *order isomorphism*.

Actually, for mapping between $\mathscr{P}A$ and $\mathscr{P}B$ where A and B are some sets (including the above considered case of funcoids from A to B) bijection and order isomorphism are essentially the same:

Proposition 15. Bijections F between sets A and B bijectively correspond to order isomorphisms f between $\mathscr{P}A$ and $\mathscr{P}B$ by the formula $f = \langle F\rangle$.

Proof. Let F is a bijection. Then $X \subseteq Y \Rightarrow \langle F\rangle X \subseteq \langle F\rangle Y$ and $\langle F^{-1}\rangle\langle F\rangle X = X$ for every sets X, $Y \in \mathscr{P}A$. Thus $f = \langle F\rangle$ is an order isomorphism.

Let now f is an order isomorphism between $\mathscr{P}A$ and $\mathscr{P}B$. Then $f(\{x\})$ is a singleton for every $x \in A$. Take $F(x)$ to the unique y such that $f(\{x\}) = \{y\}$. Obviously f is a bijection and $f = \langle F\rangle$. □

For arbitrary pointfree funcoids isomorphisms do not necessarily preserve structure. It holds only for *increasing pointfree funcoids*:

Definition 16. I call a pointfree funcoid f *increasing* iff $\langle f\rangle$ and $\langle f^{-1}\rangle$ are monotone functions.

Proposition 17. If f is an increasing isomorphism of the category of pointfree funcoids then $\langle f\rangle$ is an order isomorphism.

Proof. We have: $\langle f\rangle \circ \langle f^{-1}\rangle = \langle f \circ f^{-1}\rangle = \langle \mathrm{id}_{\mathfrak{B}}^{\mathsf{FCD}}\rangle = \mathrm{id}_{\mathfrak{B}}$ and $\langle f^{-1}\rangle \circ \langle f\rangle = \langle f^{-1} \circ f\rangle = \langle \mathrm{id}_{\mathfrak{A}}^{\mathsf{FCD}}\rangle = \mathrm{id}_{\mathfrak{A}}$. Thus $\langle f\rangle$ is a bijection.

$\langle f\rangle$ is increasing and bijective. □

Remark 18. Non-increasing isomorphisms of the category of pointfree funcoids are against sound mind, they don't preserve the structure of the source, that is for them $\langle f\rangle$ or $\langle f^{-1}\rangle$ are not order isomorphisms.

Obvious 19. Isomorphisms of **cont**(**Pos**) and **cont**(**mePos**) are order isomorphisms.

6 Direct products

[TODO: Now this section is a complete mess. Clean it up.]

Consider the category **contFcd** which is the full subcategory **cont**(**mePos**) restricted to objects which are essentially increasing pointfree funcoids.

Let $f_1 \colon Y \to X_1$ and $f_2 \colon Y \to X_2$ are morphisms of **contFcd**.

The product object is $X_1 \times^{(C)} X_2$ (cross composition product of funcoids used). It is easy to see that $X_1 \times^{(C)} X_2$ is an object of **contFcd** that is an endo-funcoid.

The morphism $f_1 \times^{(D)} f_2 \colon Y \to X_1 \times^{(C)} X_2$ is defined by the formula $(f_1 \times^{(D)} f_2) y = f_1 y \times^{\mathsf{FCD}} f_2 y$.

$f_1 \times^{(D)} f_2$ is monovalued and entirely defined because so are f_1 and f_2.

$$(f_1 \times^{(D2)} f_2) y = \bigcup \{f_1 x \times^{\mathsf{FCD}} f_2 x \mid x \in \mathrm{atoms}^{\mathfrak{A}} y\}.$$

[TODO: Is $(f_1 \times^{(D2)} f_2)$ a pointfree funcoid?]

To prove that it is really a morphism we need to show

$$\left(f_1 \times^{(D)} f_2\right) \circ Y \sqsubseteq \left(X_1 \times^{(C)} X_2\right) \circ \left(f_1 \times^{(D)} f_2\right)$$

that is (for every y)

$$\left(f_1 \times^{(D)} f_2\right) Y y \sqsubseteq \left(X_1 \times^{(C)} X_2\right)\left(f_1 \times^{(D)} f_2\right) y.$$

Really, $\left(f_1 \times^{(D)} f_2\right) Y y = f_1 Y y \times^{\mathsf{FCD}} f_2 Y y$;
$\left(X_1 \times^{(C)} X_2\right)\left(f_1 \times^{(D)} f_2\right) y = \left(X_1 \times^{(C)} X_2\right)(f_1 y \times^{\mathsf{FCD}} f_2 y) = X_1 f_1 y \times^{\mathsf{FCD}} X_2 f_2 y$;
but it is easy to show $f_1 Y y \times^{\mathsf{FCD}} f_2 Y y \sqsubseteq X_1 f_1 y \times^{\mathsf{FCD}} X_2 f_2 y$.
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I define ??
[TODO: Prove that it is a direct product in **contFcd**.]
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Bibliography

[1] Victor Porton. Pointfree funcoids. At `http://www.mathematics21.org/binaries/pointfree.pdf`.

[2] Victor Porton. Filters on posets and generalizations. *International Journal of Pure and Applied Mathematics*, 74(1):55–119, 2012.