# Algebraic General Topology. Volume 1 addons

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ABSTRACT. This file contains future addons for the free e-book "Algebraic General Topology. Volume 1", which are yet not enough ripe to be included into the book.

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CHAPTER 1

# About this document

This file contains future addons for the free e-book “Algebraic General Topology. Volume 1”, which are yet not enough ripe to be included into the book.

Theorem (including propositions, conjectures, etc.) numbers in this document start from the last theorem number in the book plus one. Theorems references inside this document are hyperlinked, but references to theorems in the book are not hyperlinked (because PDF viewer Okular 0.20.2 does not support Backward button after clicking a cross-document reference, and thus I want to avoid clicking such links).

CHAPTER 2

# Applications of algebraic general topology

## 1. "Hybrid" objects

Algebraic general topology allows to construct "hybrid" objects of "continuous" (as topological spaces) and discrete (as graphs).

Consider for example $D \sqcup T$ where $D$ is a digraph and $T$ is a topological space.

The $n$-th power $(D \sqcup T)^n$ yields an expression with $2^n$ terms. So treating $D \sqcup T$ as one object (what becomes possible using algebraic general topology) rather than the join of two objects may have an exponential benefit for simplicity of formulas.

## 2. A way to construct directed topological spaces

**2.1. Some notation.** I use $\mathcal{E}$ and $\iota$ notations from `volume-2.pdf`. FiXme: Reorder document fragments to describe it before use.

I remind that $f|_X = f \circ \mathrm{id}_X$ for binary relations, funcoids, and reloid.

$f \parallel_X = f \circ (\mathcal{E}^X)^{-1}$.

$f \square X = \mathrm{id}_X \circ f \circ \mathrm{id}_X^{-1}$.

As proved in `volume-2.pdf`, the following are bijections and moreover isomorphisms (for $R$ being either funcoids or reloids or binary relations):

1°. $\left\{ \frac{(f|_X, f\parallel_X)}{f \in R} \right\}$;
2°. $\left\{ \frac{(f \square X, \iota_X f)}{f \in R} \right\}$.

As easily follows from these isomorphisms and theorem 1100:

Proposition 1859. For funcoids, reloids, and binary relations:

1°. $f \in \mathrm{C}(\mu, \nu) \Rightarrow f \parallel_A \in \mathrm{C}(\iota_A \mu, \nu)$;
2°. $f \in \mathrm{C}'(\mu, \nu) \Rightarrow f \parallel_A \in \mathrm{C}'(\iota_A \mu, \nu)$;
3°. $f \in \mathrm{C}''(\mu, \nu) \Rightarrow f \parallel_A \in \mathrm{C}''(\iota_A \mu, \nu)$.

**2.2. Directed line and directed intervals.** Let $\mathfrak{A}$ be a poset. We will denote $\overline{\mathfrak{A}} = \mathfrak{A} \cup \{-\infty, +\infty\}$ the poset with two added elements $-\infty$ and $+\infty$, such that $+\infty$ is strictly greater than every element of $\mathfrak{A}$ and $-\infty$ is strictly less.

FiXme: Generalize from $\mathbb{R}$ to a wider class of posets.

Definition 1860. For an element $a$ of a poset $\mathfrak{A}$

1°. $J_{\geq}(a) = \left\{ \frac{x \in \mathfrak{A}}{x \geq a} \right\}$;
2°. $J_{>}(a) = \left\{ \frac{x \in \mathfrak{A}}{x > a} \right\}$;
3°. $J_{\leq}(a) = \left\{ \frac{x \in \mathfrak{A}}{x \leq a} \right\}$;
4°. $J_{<}(a) = \left\{ \frac{x \in \mathfrak{A}}{x < a} \right\}$;
5°. $J_{\neq}(a) = \left\{ \frac{x \in \mathfrak{A}}{x \neq a} \right\}$.

Definition 1861. Let $a$ be an element of a poset $\mathfrak{A}$.

1°. $\Delta(a) = \bigsqcap^{\mathscr{F}} \left\{ \frac{]x;y[}{x,y \in \overline{\mathfrak{A}}, x < a \wedge y > a} \right\}$;
2°. $\Delta_{\geq}(a) = \bigsqcap^{\mathscr{F}} \left\{ \frac{[a;y[}{y \in \overline{\mathfrak{A}}, y > a} \right\}$;

3°. $\Delta_{>}(a)=\prod^{\mathscr{F}}\left\{\frac{]a;y[}{y\in\overline{\mathfrak{A}},x<a\wedge y>a}\right\}$;
4°. $\Delta_{\leq}(a)=\prod^{\mathscr{F}}\left\{\frac{]x;a]}{x\in\overline{\mathfrak{A}},x<a}\right\}$;
5°. $\Delta_{<}(a)=\prod^{\mathscr{F}}\left\{\frac{]x;a[}{x\in\overline{\mathfrak{A}},x<a}\right\}$;
6°. $\Delta_{\neq}(a)=\Delta(a)\setminus\{a\}$.

Obvious 1862.
1°. $\Delta_{\geq}(a)=\Delta(a)\sqcap^{\mathscr{F}}@J_{\geq}(a)$;
2°. $\Delta_{>}(a)=\Delta(a)\sqcap^{\mathscr{F}}@J_{>}(a)$;
3°. $\Delta_{\leq}(a)=\Delta(a)\sqcap^{\mathscr{F}}@J_{\leq}(a)$;
4°. $\Delta_{<}(a)=\Delta(a)\sqcap^{\mathscr{F}}@J_{<}(a)$;
5°. $\Delta_{\neq}(a)=\Delta(a)\sqcap^{\mathscr{F}}@J_{\neq}(a)$.

Definition 1863. Given a partial order $\mathfrak{A}$ and $x\in\mathfrak{A}$, the following defines complete funcoids:
1°. $\langle|\mathfrak{A}|\rangle^{*}\{x\}=\Delta(x)$;
2°. $\langle|\mathfrak{A}|_{\geq}\rangle^{*}\{x\}=\Delta_{\geq}(x)$;
3°. $\langle|\mathfrak{A}|_{>}\rangle^{*}\{x\}=\Delta_{>}(x)$;
4°. $\langle|\mathfrak{A}|_{\leq}\rangle^{*}\{x\}=\Delta_{\leq}(x)$;
5°. $\langle|\mathfrak{A}|_{<}\rangle^{*}\{x\}=\Delta_{<}(x)$;
6°. $\langle|\mathfrak{A}|_{\neq}\rangle^{*}\{x\}=\Delta_{\neq}(x)$.

Proposition 1864. The complete funcoid corresponding to the order topology[1] is equal to $|\mathfrak{A}|$.

Proof. Because every open set is a finite union of open intervals, the complete funcoid $f$ corresponding to the order topology is described by the formula: $\langle f\rangle^{*}\{x\}=\prod^{\mathscr{F}}\left\{\frac{]a;b[}{a,b\in\overline{\mathfrak{A}},a<x\wedge b>x}\right\}=\Delta(x)=\langle|\mathfrak{A}|\rangle^{*}\{x\}$. Thus $f=|\mathfrak{A}|$. □

Exercise 1865. Show that $|\mathfrak{A}|_{\geq}$ (in general) is not the same as "right order topology"[2].

Proposition 1866.
1°. $\left\langle|\mathfrak{A}|_{\geq}^{-1}\right\rangle^{*}@X=@\left\{\frac{a\in\mathfrak{A}}{\forall y\in\overline{\mathfrak{A}}:(y>a\Rightarrow X\cap[a;y[\neq\emptyset)}\right\}$;
2°. $\left\langle|\mathfrak{A}|_{>}^{-1}\right\rangle^{*}@X=@\left\{\frac{a\in\mathfrak{A}}{\forall y\in\overline{\mathfrak{A}}:(y>a\Rightarrow X\cap]a;y[\neq\emptyset)}\right\}$;
3°. $\left\langle|\mathfrak{A}|_{\leq}^{-1}\right\rangle^{*}@X=@\left\{\frac{a\in\mathfrak{A}}{\forall x\in\overline{\mathfrak{A}}:(x<a\Rightarrow X\cap]x;a]\neq\emptyset)}\right\}$;
4°. $\left\langle|\mathfrak{A}|_{<}^{-1}\right\rangle^{*}@X=@\left\{\frac{a\in\mathfrak{A}}{\forall x\in\overline{\mathfrak{A}}:(x<a\Rightarrow X\cap]x;a[\neq\emptyset)}\right\}$.

Proof. $a\in\left\langle|\mathfrak{A}|_{\geq}^{-1}\right\rangle^{*}@X\Leftrightarrow@\{a\}\not\asymp\left\langle|\mathfrak{A}|_{\geq}^{-1}\right\rangle^{*}@X\Leftrightarrow\langle|\mathfrak{A}|_{\geq}\rangle^{*}@\{a\}\not\asymp@X\Leftrightarrow\Delta_{\geq}(a)\not\asymp@X\Leftrightarrow\forall y\in\overline{\mathfrak{A}}:(y>a\Rightarrow X\cap[a;y[\neq\emptyset)$.

$a\in\left\langle|\mathfrak{A}|_{>}^{-1}\right\rangle^{*}@X\Leftrightarrow@\{a\}\not\asymp\left\langle|\mathfrak{A}|_{>}^{-1}\right\rangle^{*}@X\Leftrightarrow\langle|\mathfrak{A}|_{>}\rangle^{*}@\{a\}\not\asymp@X\Leftrightarrow\Delta_{>}(a)\not\asymp@X\Leftrightarrow\forall y\in\overline{\mathfrak{A}}:(y>a\Rightarrow X\cap]a;y[\neq\emptyset)$.

The rest follows from duality. □

Remark 1867. On trivial ultrafilters these obviously agree:
1°. $\langle|\mathbb{R}|_{\geq}\rangle^{*}\{x\}=\langle|\mathbb{R}|\sqcap\geq\rangle^{*}\{x\}$;
2°. $\langle|\mathbb{R}|_{>}\rangle^{*}\{x\}=\langle|\mathbb{R}|\sqcap>\rangle^{*}\{x\}$;
3°. $\langle|\mathbb{R}|_{\leq}\rangle^{*}\{x\}=\langle|\mathbb{R}|\sqcap\leq\rangle^{*}\{x\}$;
4°. $\langle|\mathbb{R}|_{<}\rangle^{*}\{x\}=\langle|\mathbb{R}|\sqcap<\rangle^{*}\{x\}$.

[1] See Wikipedia for a definition of "Order topology".
[2] See Wikipedia

COROLLARY 1868.

1°. $|\mathbb{R}|_{\geq} = \operatorname{Compl}(|\mathbb{R}| \sqcap \geq)$;
2°. $|\mathbb{R}|_{>} = \operatorname{Compl}(|\mathbb{R}| \sqcap >)$;
3°. $|\mathbb{R}|_{\leq} = \operatorname{Compl}(|\mathbb{R}| \sqcap \leq)$;
4°. $|\mathbb{R}|_{<} = \operatorname{Compl}(|\mathbb{R}| \sqcap <)$.

OBVIOUS 1869. FiXme: also what is the values of $\setminus$ operation

1°. $|\mathbb{R}|_{\geq} = |\mathbb{R}|_{>} \sqcup 1$;
2°. $|\mathbb{R}|_{\leq} = |\mathbb{R}|_{<} \sqcup 1$.

## 3. Some inequalities

FiXme: Define the ultrafilter "at the left" and "at the right" of a real number. Also define "convergent ultrafilter".

Denote $\Delta_{+\infty} = \prod_{x\in\mathbb{R}} ]x; +\infty[$ and $\Delta_{-\infty} = \prod_{x\in\mathbb{R}} ] - \infty; x[$.

The following proposition calculates $\langle\geq\rangle x$ and $\langle>\rangle x$ for all kinds of ultrafilters on $\mathbb{R}$:

PROPOSITION 1870.

1°. $\langle\geq\rangle\{\alpha\} = [\alpha; +\infty[$ and $\langle>\rangle\{\alpha\} = ]\alpha; +\infty[$.
2°. $\langle\geq\rangle x = \langle>\rangle x = ]\alpha; +\infty[$ for ultrafilter $x$ at the right of a number $\alpha$.
3°. $\langle\geq\rangle x = \langle>\rangle x = \Delta_{<}(\alpha) \sqcup [\alpha; +\infty[ = \Delta_{\leq}(\alpha) \sqcup ]\alpha; +\infty[$ for ultrafilter $x$ at the left of a number $\alpha$.
4°. $\langle\geq\rangle x = \langle>\rangle x = \Delta_{+\infty}$ for ultrafilter $x$ at positive infinity.
5°. $\langle\geq\rangle x = \langle>\rangle x = \mathbb{R}$ for ultrafilter $x$ at negative infinity.

PROOF.
1°. Obvious.
2°.

$$\langle\geq\rangle x = \prod^{\mathscr{F}}_{X\in\operatorname{up} x} \langle\geq\rangle(X \sqcap ]\alpha; +\infty[) = \prod^{\mathscr{F}}_{X\in\operatorname{up} x} ]\alpha; +\infty[ = ]\alpha; +\infty[;$$

$$\langle>\rangle x = \prod^{\mathscr{F}}_{X\in\operatorname{up} x} \langle>\rangle(X \sqcap ]\alpha; +\infty[) = \prod^{\mathscr{F}}_{X\in\operatorname{up} x} ]\alpha; +\infty[ = ]\alpha; +\infty[.$$

3°. $\Delta_{<}(\alpha) \sqcup [\alpha; +\infty[ = \Delta_{\leq}(\alpha) \sqcup ]\alpha; +\infty[$ is obvious.

$$\langle>\rangle x = \prod^{\mathscr{F}}_{X\in\operatorname{up} x} \langle>\rangle X \sqsupseteq \prod^{\mathscr{F}}_{X\in\operatorname{up} x} (\Delta_{<}(\alpha) \sqcup ]\alpha; +\infty[) = \Delta_{<}(\alpha) \sqcup ]\alpha; +\infty[$$

but $\langle\geq\rangle x \sqsubseteq \Delta_{<}(\alpha) \sqcup [\alpha; +\infty[$ is obvious. It remains to take into account that $\langle>\rangle x \sqsubseteq \langle\geq\rangle x$.

4°. $\langle\geq\rangle x = \prod^{\mathscr{F}}_{X\in\operatorname{up} x} \langle\geq\rangle X = \prod^{\mathscr{F}}_{X\in\operatorname{up} x, \inf X\in X} \langle\geq\rangle(X \sqcap ]\alpha; +\infty[) = \prod^{\mathscr{F}}_{X\in\operatorname{up} x} [\inf X; +\infty[ = \prod^{\mathscr{F}}_{x>\alpha} [x; +\infty[ = \Delta_{+\infty}$; $\langle>\rangle x = \prod^{\mathscr{F}}_{X\in\operatorname{up} x} \langle>\rangle X = \prod^{\mathscr{F}}_{X\in\operatorname{up} x, \inf X\in X} \langle>\rangle(X \sqcap ]\alpha; +\infty[) = \prod^{\mathscr{F}}_{X\in\operatorname{up} x} ]\inf X; +\infty[ = \prod^{\mathscr{F}}_{x>\alpha} [x; +\infty[ = \Delta_{+\infty}$.

5°. $\langle\geq\rangle x \sqsupseteq \langle>\rangle x = \prod^{\mathscr{F}}_{X\in\operatorname{up} x} \langle>\rangle X$ but $\langle>\rangle X = ] - \infty; +\infty[$ for $X \in \operatorname{up} x$ because $X$ has arbitrarily small elements.

□

LEMMA 1871. $\langle|\mathbb{R}|\rangle x \sqsubseteq \langle>\rangle x = \langle\geq\rangle x$ for every nontrivial ultrafilter $x$.

PROOF. $\langle>\rangle x = \langle\geq\rangle x$ follows from the previous proposition.
$\langle|\mathbb{R}|\rangle x = \prod_{X\in\operatorname{up} x} \langle|\mathbb{R}|\rangle X = \prod_{X\in\operatorname{up} x} \bigsqcup_{y\in X} \Delta(y)$.
Consider cases:

$x$ is an ultrafilter at the right of some number $\alpha$.
$\langle|\mathbb{R}|\rangle x = \bigsqcap_{X\in\operatorname{up}x}\bigsqcup_{y\in X\sqcap]\alpha;+\infty[}\Delta(y) \sqsubseteq ]\alpha;+\infty[ = \langle\geq\rangle x$ because $\bigsqcup_{y\in X\sqcap]\alpha;+\infty[}\Delta(y)\sqsubseteq]\alpha;+\infty[$.

$x$ is an ultrafilter at the left of some number $\alpha$.
$\langle|\mathbb{R}|\rangle x \sqsubseteq \Delta(\alpha)$ is obvious. But $\langle\geq\rangle x \sqsupseteq \Delta(\alpha)$.

$x$ is an ultrafilter at positive infinity.
$\langle|\mathbb{R}|\rangle x \sqsubseteq \Delta_{+\infty}$ is obvious. But $\langle\geq\rangle x = \Delta_{+\infty}$.

$x$ is an ultrafilter at negative infinity.
Because $\langle\geq\rangle x = \mathbb{R}$.

□

Corollary 1872. $\langle|\mathbb{R}|\sqcap\geq\rangle x = \langle|\mathbb{R}|\rangle x$ for every nontrivial ultrafilter $x$.

Proof. $\langle|\mathbb{R}|\sqcap\geq\rangle x = \langle|\mathbb{R}|\rangle\sqcap\langle\geq\rangle x = \langle|\mathbb{R}|\rangle x$. □

So $\langle|\mathbb{R}|\sqcap\geq\rangle$ and $\langle|\mathbb{R}|\rangle$ agree on all ultrafilters except trivial ones.

Proposition 1873. $|\mathbb{R}|_>\sqcap > = |\mathbb{R}|_>\sqcap\geq = |\mathbb{R}|_>$.

Proof. $|\mathbb{R}|_>\sqsubseteq >$ because $\langle|\mathbb{R}|_>\rangle^* x\sqsubseteq\langle>\rangle^* x$ and $|\mathbb{R}|_>$ is a complete funcoid. □

Lemma 1874. $\langle|\mathbb{R}|_>\rangle x\sqsubset\langle|\mathbb{R}|_\geq\rangle x$ for a nontrivial ultrafilter $x$.

Proof. It enough to prove $\langle|\mathbb{R}|_>\rangle x\neq\langle|\mathbb{R}|_\geq\rangle x$.
Take $x$ be an ultrafilter with limit point $0$ on $\operatorname{im}z$ where $z$ is the sequence $n\mapsto\frac{1}{n}$.

$$\langle|\mathbb{R}|_>\rangle x\sqsubseteq\langle|\mathbb{R}|_>\rangle^*\operatorname{im}z = \bigsqcup_{n\in\operatorname{im}z}\Delta_>\left(\frac{1}{n}\right)\sqsubseteq\bigsqcup_{n\in\operatorname{im}z}\left]\frac{1}{n};\frac{1}{n-1}-\frac{1}{n}\right[\asymp\operatorname{im}z.$$

Thus $\langle|\mathbb{R}|_>\rangle x\asymp\operatorname{im}z$. But $\langle|\mathbb{R}|_\geq\rangle x\sqsubseteq\langle=\rangle x\not\asymp\operatorname{im}z$. □

Corollary 1875. $|\mathbb{R}|_>\sqsubset|\mathbb{R}|_\geq$.

Proposition 1876. $|\mathbb{R}|_>\sqsubset|\mathbb{R}|_\geq\sqcap >$.

Proof. It's enough to prove $|\mathbb{R}|_>\neq|\mathbb{R}|_\geq\sqcap >$.
Really, $\langle|\mathbb{R}|_\geq\sqcap >\rangle x = \langle|\mathbb{R}|_\geq\rangle x\neq\langle|\mathbb{R}|_>\rangle x$ (lemma). □

Proposition 1877.
1°. $|\mathbb{R}|_\geq\circ|\mathbb{R}|_\geq = |\mathbb{R}|_\geq$;
2°. $|\mathbb{R}|_>\circ|\mathbb{R}|_> = |\mathbb{R}|_>$;
3°. $|\mathbb{R}|_\geq\circ|\mathbb{R}|_> = |\mathbb{R}|_>$;
4°. $|\mathbb{R}|_>\circ|\mathbb{R}|_\geq = |\mathbb{R}|_>$.

Proof. ?? □

Conjecture 1878.
1°. $(|\mathbb{R}|\sqcap\geq)\circ(|\mathbb{R}|\sqcap\geq) = |\mathbb{R}|\sqcap\geq$.
2°. $(|\mathbb{R}|\sqcap >)\circ(|\mathbb{R}|\sqcap >) = |\mathbb{R}|\sqcap >$.

## 4. Continuity

I will say that a property holds on a filter $\mathcal{A}$ iff there is $A\in\operatorname{up}\mathcal{A}$ on which the property holds.

FiXme: $f\in\mathrm{C}(A,B)\wedge f\in\mathrm{C}(\iota_A|\mathbb{R}|_\geq,\iota_B|\mathbb{R}|_\geq)\Leftrightarrow(f,f)\in\mathrm{C}((A,\iota_A|\mathbb{R}|_\geq),(B,\iota_B|\mathbb{R}|_\geq))$

Lemma 1879. Let function $f:A\to B$ where $A,B\in\mathscr{P}\mathbb{R}$ and $A$ is connected.

1°. $f$ is monotone and $f \in \mathrm{C}(A,B)$ iff $f \in \mathrm{C}(A,B) \cap \mathrm{C}(\iota_A|\mathbb{R}|_\geq, \iota_B|\mathbb{R}|_\geq)$ iff $f \in \mathrm{C}(A,B) \cap \mathrm{C}(\iota_A|\mathbb{R}|_>, \iota_B|\mathbb{R}|_\geq)$ iff $f \in \mathrm{C}(\iota_A|\mathbb{R}|_\geq, \iota_B|\mathbb{R}|_\geq) \cap \mathrm{C}(\iota_A|\mathbb{R}|_\leq, \iota_B|\mathbb{R}|_\leq)$.

2°. $f$ is strictly monotone and $\in \mathrm{C}(A,B)$ iff $f \in \mathrm{C}(A,B) \cap \mathrm{C}(\iota_A|\mathbb{R}|_>, \iota_B|\mathbb{R}|_>)$ iff $f \in \mathrm{C}(\iota_A|\mathbb{R}|_>, \iota_B|\mathbb{R}|_>) \cap \mathrm{C}(\iota_A|\mathbb{R}|_<, \iota_B|\mathbb{R}|_<)$.

FiXme: Generalize for arbitrary posets. FiXme: Generalize for $f$ being a funcoid.

Proof. Because $f$ is continuous, we have $\langle f \circ \iota_A|\mathbb{R}|\rangle^* \{x\} \sqsubseteq \langle \iota_B|\mathbb{R}| \circ f\rangle^* \{x\}$ that is $\langle f\rangle^* \Delta(x) \sqsubseteq \Delta(f(x))$ for every $x$.

If $f$ is monotone, we have $\langle f\rangle^* \Delta_\geq(x) \sqsubseteq [f(x);\infty[$. Thus $\langle f\rangle^* \Delta_\geq(x) \sqsubseteq \Delta_\geq(f(x))$, that is $\langle f \circ \iota_A|\mathbb{R}|_\geq\rangle^* \{x\} \sqsubseteq \langle \iota_B|\mathbb{R}|_\geq \circ f\rangle^* \{x\}$, thus $f \in \mathrm{C}(\iota_A|\mathbb{R}|_\geq, \iota_B|\mathbb{R}|_\geq)$.

If $f$ is strictly monotone, we have $\langle f\rangle^* \Delta_>(x) \sqsubseteq ]f(x);\infty[$. Thus $\langle f\rangle^* \Delta_>(x) \sqsubseteq \Delta_>(f(x))$, that is $\langle f \circ \iota_A|\mathbb{R}|_>\rangle^* \{x\} \sqsubseteq \langle \iota_B|\mathbb{R}|_> \circ f\rangle^* \{x\}$, thus $f \in \mathrm{C}(\iota_A|\mathbb{R}|_>, \iota_B|\mathbb{R}|_>)$.

Let now $f \in \mathrm{C}(\iota_A|\mathbb{R}|_\geq, \iota_B|\mathbb{R}|_\geq)$.

Take any $a \in A$ and let $c = \left\{\frac{b\in B}{b \geq a, \forall x \in [a;b[: f(x) \geq f(a)}\right\}$. It's enough to prove that $c$ is the right endpoint (finite or infinite) of $A$.

Indeed by continuity $f(a) \leq f(c)$ and if $c$ is not already the right endpoint of $A$, then there is $b' > c$ such that $\forall x \in [c;b'[: f(x) \geq f(c)$. So we have $\forall x \in [a;b'[: f(x) \geq f(c)$ what contradicts to the above.

So $f$ is monotone on the entire $A$.

$f \in \mathrm{C}(\iota_A|\mathbb{R}|_\geq, \iota_B|\mathbb{R}|_\geq) \Rightarrow f \in \mathrm{C}(\iota_A|\mathbb{R}|_>, \iota_B|\mathbb{R}|_\geq)$ is obvious. Reversely $f \in \mathrm{C}(\iota_A|\mathbb{R}|_>, \iota_B|\mathbb{R}|_\geq) \Rightarrow f \circ \iota_A|\mathbb{R}|_> \sqsubseteq \iota_B|\mathbb{R}|_\geq \circ f \Leftrightarrow \forall x \in \mathbb{R} : \langle f\rangle\langle \iota_A|\mathbb{R}|_>\rangle^* \{x\} \sqsubseteq \langle \iota_B|\mathbb{R}|_\geq\rangle^* \langle f\rangle^* \{x\} \Leftrightarrow \forall x \in \mathbb{R} : \langle f\rangle \Delta_>(x) \sqsubseteq \Delta_\geq f(x) \Leftrightarrow \forall x \in \mathbb{R} : \langle f\rangle \Delta_>(x) \sqcup \{f(x)\} \sqsubseteq \Delta_\geq f(x) \Leftrightarrow \forall x \in \mathbb{R} : \langle f\rangle \Delta_>(x) \sqcup \{x\} \sqsubseteq \Delta_\geq f(x) \Leftrightarrow \forall x \in \mathbb{R} : \langle f\rangle \Delta_\geq(x) \sqsubseteq \Delta_\geq f(x) \Leftrightarrow \forall x \in \mathbb{R} : \langle f\rangle\langle \iota_A|\mathbb{R}|_\geq\rangle^* \{x\} \sqsubseteq \langle \iota_B|\mathbb{R}|_\geq\rangle^* \langle f\rangle^* \{x\} \Leftrightarrow \forall x \in \mathbb{R} : f \circ \iota_A|\mathbb{R}|_\geq \sqsubseteq \iota_B|\mathbb{R}|_\geq \circ f \Leftrightarrow f \in \mathrm{C}(\iota_A|\mathbb{R}|_\geq, \iota_B|\mathbb{R}|_\geq)$.

Let $f \in \mathrm{C}(\iota_A|\mathbb{R}|_>, \iota_B|\mathbb{R}|_>)$. Then $f \in \mathrm{C}(\iota_A|\mathbb{R}|_>, \iota_B|\mathbb{R}|_\geq)$ and thus it is monotone. We need to prove that $f$ is strictly monotone. Suppose the contrary. Then there is a nonempty interval $[p;q] \subseteq A$ such that $f$ is constant on this interval. But this is impossible because $f \in \mathrm{C}(\iota_A|\mathbb{R}|_>, \iota_B|\mathbb{R}|_>)$.

Prove that $f \in \mathrm{C}(\iota_A|\mathbb{R}|_\geq, \iota_B|\mathbb{R}|_\geq) \cap \mathrm{C}(\iota_A|\mathbb{R}|_\leq, \iota_B|\mathbb{R}|_\leq)$ implies $f \in \mathrm{C}(A,B)$. Really, it implies $\langle f\rangle \Delta_\leq(x) \sqsubseteq \Delta_\leq(fx)$ and $\langle f\rangle \Delta_\geq(x) \sqsubseteq \Delta_\geq(fx)$ thus $\langle f\rangle \Delta(x) = \langle f\rangle(\Delta_\leq(x) \sqcup \{x\} \sqcup \Delta_\geq(x)) \subseteq \Delta_\leq f(x) \sqcup \{f(x)\} \sqcup \Delta_\geq f(x) = \Delta(f(x))$.

Prove that $f \in \mathrm{C}(\iota_A|\mathbb{R}|_>, \iota_B|\mathbb{R}|_>) \cap \mathrm{C}(\iota_A|\mathbb{R}|_<, \iota_B|\mathbb{R}|_<)$ $f \in \mathrm{C}(A,B)$. Really, it implies $\langle f\rangle \Delta_<(x) \sqsubseteq \Delta_<(fx)$ and $\langle f\rangle \Delta_>(x) \sqsubseteq \Delta_>(fx)$ thus $\langle f\rangle \Delta(x) = \langle f\rangle(\Delta_<(x) \sqcup \{x\} \sqcup \Delta_>(x)) \subseteq \Delta_< f(x) \sqcup \{f(x)\} \sqcup \Delta_> f(x) = \Delta(f(x))$. □

Theorem 1880. Let function $f : A \to B$ where $A, B \in \mathscr{P}\mathbb{R}$.

1°. $f$ is locally monotone and $f \in \mathrm{C}(A,B)$ iff $f \in \mathrm{C}(A,B) \cap \mathrm{C}(\iota_A|\mathbb{R}|_\geq, \iota_B|\mathbb{R}|_\geq)$ iff $f \in \mathrm{C}(A,B) \cap \mathrm{C}(\iota_A|\mathbb{R}|_>, \iota_B|\mathbb{R}|_\geq)$ iff $f \in \mathrm{C}(\iota_A|\mathbb{R}|_\geq, \iota_B|\mathbb{R}|_\geq) \cap \mathrm{C}(\iota_A|\mathbb{R}|_\leq, \iota_B|\mathbb{R}|_\leq)$.

2°. $f$ is locally strictly monotone and $\in \mathrm{C}(A,B)$ iff $f \in \mathrm{C}(A,B) \cap \mathrm{C}(\iota_A|\mathbb{R}|_>, \iota_B|\mathbb{R}|_>)$ iff $f \in \mathrm{C}(\iota_A|\mathbb{R}|_>, \iota_B|\mathbb{R}|_>) \cap \mathrm{C}(\iota_A|\mathbb{R}|_<, \iota_B|\mathbb{R}|_<)$.

Proof. By the lemma it is (strictly) monotone on each connected component. □

See also related math.SE questions:

1°. http://math.stackexchange.com/q/1473668/4876
2°. http://math.stackexchange.com/a/1872906/4876
3°. http://math.stackexchange.com/q/1875975/4876

**4.1. Directed topological spaces.** Directed topological spaces are defined at
http://ncatlab.org/nlab/show/directed+topological+space

Definition 1881. A *directed topological space* (or *d-space* for short) is a pair $(X,d)$ of a topological space $X$ and a set $d\subseteq \mathrm{C}([0;1],X)$ (called *directed paths* or *d-paths*) of paths in $X$ such that

1°. (constant paths) every constant map $[0;1]\to X$ is directed;
2°. (reparameterization) $d$ is closed under composition with increasing continuous maps $[0;1]\to[0;1]$;
3°. (concatenation) $d$ is closed under path-concatenation: if the d-paths $a$, $b$ are consecutive in $X$ ($a(1)=b(0)$), then their ordinary concatenation $a+b$ is also a d-path

$$(a+b)(t)=a(2t),\ \text{if}\, 0\leq t\leq \frac{1}{2},$$
$$(a+b)(t)=b(2t-1),\ \text{if}\, \frac{1}{2}\leq t\leq 1.$$

I propose a new way to construct a directed topological space. My way is more geometric/topological as it does not involve dealing with particular paths.

Definition 1882. Let $T$ be the complete endofuncoid corresponding to a topological space and $\nu\sqsubseteq T$ be its "subfuncoid". The d-space $(\mathrm{dir})(T,\nu)$ induced by the pair $(T,\nu)$ consists of $T$ and paths $f\in\mathrm{C}([0;1],T)\cap\mathrm{C}(|[0;1]|_{\geq},\nu)$ such that $f(0)=f(1)$.

Proposition 1883. It is really a d-space.

Proof. Every d-path is continuous.

Constant path are d-paths because $\nu$ is reflexive.

Every reparameterization is a d-path because they are $\mathrm{C}(|[0;1]|_{\geq},\nu)$ and we can apply the theorem about composition of continuous functions.

Every concatenation is a d-path. Denote $f_0=\lambda t\in[0;\frac{1}{2}]:a(2t)$ and $f_1=\lambda t\in[\frac{1}{2};1]:b(2t-1)$. Obviously $f_0,f_1\in\mathrm{C}([0;1],\mu)\cap\mathrm{C}(|[0;1]|_{\geq},\nu)$. Then we conclude that $a+b=f_1\sqcup f_1$ is in $f_0,f_1\in\mathrm{C}([0;1],\mu)\cap\mathrm{C}(|[0;1]|_{\geq},\nu)$ using the fact that the operation $\circ$ is distributive over $\sqcup$. $\square$

Below we show that not every d-space is induced by a pair of an endofuncoid and its subfuncoid. But are d-spaces not represented this way good anything except counterexamples?

Let now we have a d-space $(X,d)$. Define funcoid $\nu$ corresponding to the d-space by the formula $\nu=\bigsqcup_{a\in d}(a\circ|\mathbb{R}|_{\geq}\circ a^{-1})$.

Example 1884. The two directed topological spaces, constructed from a fixed topological space and two different reflexive funcoids, are the same.

Proof. Consider the indiscrete topology $T$ on $\mathbb{R}$ and the funcoids $1^{\mathsf{FCD}(\mathbb{R},\mathbb{R})}$ and $1^{\mathsf{FCD}(\mathbb{R},\mathbb{R})}\sqcup(\{0\}\times^{\mathsf{FCD}}\Delta_{\geq})$. The only d-paths in both these settings are constant functions. $\square$

Example 1885. A d-space is not determined by the induced funcoid.

Proof. The following a d-space induces the same funcoid as the d-space of all paths on the plane.

Consider a plane $\mathbb{R}^2$ with the usual topology. Let d-paths be paths lying inside a polygonal chain (in the plane). $\square$

CONJECTURE 1886. A d-path $a$ is determined by the funcoids (where $x$ spans $[0;1]$)

$$(\lambda t \in \mathbb{R} : a(x+t))|_{\Delta(0)}.$$

## 5. A way to construct directed topological spaces

FiXme: Should include definition of directed topological space.

Directed topological spaces are defined at
http://ncatlab.org/nlab/show/directed+topological+space

I propose a new way to construct a directed topological space. My way is more geometric/topological as it does not involve dealing with particular paths.

CONJECTURE 1887. Every directed topological space can be constructed in the below described way.

Consider topological space $T$ and its subfuncoid $F$ (that is $F$ is a funcoid which is less that $T$ in the order of funcoids). Note that in our consideration $F$ is an endofuncoid (its source and destination are the same).

Then a directed path from point $A$ to point $B$ is defined as a continuous function $f$ from $[0;1]$ to $F$ such that $f(0) = A$ and $f(1) = B$. FiXme: Specify whether the interval $[0;1]$ is treated as a proximity, pretopology, or preclosure.

Because $F$ is less that $T$, we have that every directed path is a path.

CONJECTURE 1888. The two directed topological spaces, constructed from a fixed topological space and two different funcoids, are different.

For a counter-example of (which of the two?) the conjecture consider funcoid $T \sqcap (\mathbb{Q} \times^{\mathsf{FCD}} \mathbb{Q})$ where $T$ is the usual topology on real line.We need to consider stability of existence and uniqueness of a path under transformations of our funcoid and under transformations of the vector field. Can this be a step to solve Navier-Stokes existence and smoothness problems?

## 6. Integral curves

We will consider paths in a normed vector space $V$.

DEFINITION 1889. Let $D$ be a connected subset of $\mathbb{R}$. A *path* is a function $D \to V$.

Let $d$ be a vector field in a normed vector space $V$.

DEFINITION 1890. *Integral curve* of a vector field $d$ is a differentiable function $f : D \to V$ such that $f'(t) = d(f(t))$ for every $t \in D$.

DEFINITION 1891. The definition of *right side integral curve* is the above definition with right derivative of $f$ instead of derivative $f'$. *Left side integral curve* is defined similarly.

### 6.1. Path reparameterization.

$C^1$ is a function which has continuous derivative on every point of the domain.

By $D^1$ I will denote a $C^1$ function whose derivative is either nonzero at every point or is zero everywhere.

DEFINITION 1892. A *reparameterization* of a $C^1$ path is a bijective $C^1$ function $\phi : D \to D$ such that $\phi'(t) > 0$. A curve $f_2$ is called a reparametrized curve $f_1$ if there is a reparameterization $\phi$ such that $f_2 = f_1 \circ \phi$.

It is well known that this defines an equivalence relation of functions.

PROPOSITION 1893. Reparameterization of $D^1$ function is $D^1$.

PROOF. If the function has zero derivative, it is obvious.

Let $f_1$ has everywhere nonzero derivative. Then $f_2'(t) = f_1'(\phi(t))\phi'(t)$ what is trivially nonzero. □

DEFINITION 1894. Vectors $p$ and $q$ have the *same direction* ($p \upuparrows q$) iff there exists a strictly positive real $c$ such that $p = cq$.

OBVIOUS 1895. Being same direction is an equivalence relation.

OBVIOUS 1896. The only vector with the same direction as the zero vector is zero vector.

THEOREM 1897. A $D^1$ function $y$ is some reparameterization of a $D^1$ integral curve $x$ of a continuous vector field $d$ iff $y'(t) \upuparrows d(y(t))$ that is vectors $y'(t)$ and $d(y(t))$ have the same direction (for every $t$).

PROOF. If $y$ is a reparameterization of $x$, then $y(t) = x(\phi(t))$. Thus $y'(t) = x'(\phi(t))\phi'(t) = d(x(\phi(t)))\phi'(t) = d(y(t))\phi'(t)$. So $y'(t) \upuparrows d(y(t))$ because $\phi'(t) > 0$.

Let now $x'(t) \upuparrows d(x(t))$ that is that is there is a strictly positive function $c(t)$ such that $x'(t) = c(t)d(x(t))$.

If $x'(t)$ is zero everywhere, then $d(x(t)) = 0$ and thus $x'(t) = d(x(t))$ that is $x$ is an Integral curve. Note that $y$ is a reparameterization of itself.

We can assume that $x'(t) \neq 0$ everywhere. Then $F(x(t)) \neq 0$. We have that $c(t) = \frac{||x'(t)||}{||d(x(t))||}$ is a continuous function. FiXme: Does it work for non-normed spaces?

Let $y(u(t)) = x(t)$, where

$$u(t) = \int_0^t c(s)ds,$$

which is defined and finite because $c$ is continuous and monotone (thus having inverse defined on its image) because $c$ is positive.

Then

$$\begin{aligned} y'(u(t))u'(t) &= x'(t) \\ &= c(t)d(x(t)), \text{ so} \\ y'(u(t))c(t) &= c(t)d(y(u(t))) \\ y'(u(t)) &= d(y(u(t))) \end{aligned}$$

and letting $s = u(t)$ we have $y'(s) = d(y(s))$ for a reparameterization $y$ of $x$. □

**6.2. Vector space with additional coordinate.** Consider the normed vector space with additional coordinate $t$.

Our vector field $d(t)$ induces vector field $\hat{d}(t, v) = (1, d(v))$ in this space. Also $\hat{f}(t) = (t, f(t))$.

PROPOSITION 1898. Let $f$ be a $D^1$ function. $f$ is an integral curve of $d$ iff $\hat{f}$ is a reparametrized integral curve of $\hat{d}$.

PROOF. First note that $\hat{f}$ always has a nonzero derivative. $\hat{f}'(t) \upuparrows \hat{d}(\hat{f}(t)) \Leftrightarrow (1, f'(t)) \upuparrows (1, d(f(t))) \Leftrightarrow f'(t) = d(f(t))$. □

Thus we have reduced (for $D^1$ paths) being an integral curve to being a reparametrized integral curve. We will also describe being a reparametrized integral curve topologically (through funcoids).

**6.3. Topological description of $C^1$ curves.** Explicitly construct this funcoid as follows:

$R(d,\phi) = \left\{ \frac{v\in V}{\widehat{vd}<\phi, v\neq 0} \right\}$ for $d \neq 0$ and $R(0,\phi) = \{0\}$, where $\widehat{ab}$ is the angle between the vectors $a$ and $b$, for a direction $d$ and an angle $\phi$.

DEFINITION 1899. $W(d) = \prod^{\mathsf{RLD}} \left\{ \frac{R(d,\phi)}{\phi\in\mathbb{R},\phi>0} \right\} \sqcap \prod^{\mathsf{RLD}}_{r>0} B_r(0)$. FiXme: This is defined for infinite dimensional case. FiXme: Consider also FCD instead of RLD.

PROPOSITION 1900. For finite dimensional case $\mathbb{R}^n$ we have $W(d) = \prod^{\mathsf{RLD}} \left\{ \frac{R(d,\phi)}{\phi\in\mathbb{R},\phi>0} \right\} \sqcap \Delta^{(\mathsf{RLD})n}$ where

$$\Delta^{(\mathsf{RLD})n} = \underbrace{\Delta \times^{\mathsf{RLD}} \cdots \times^{\mathsf{RLD}} \Delta}_{n \text{ times}}.$$

PROOF. ?? □

Finally our funcoids are the complete funcoids $Q_+$ and $Q_-$ described by the formulas

$$\langle Q_+\rangle^* @\{p\} = \langle p+\rangle W(d(p)) \quad \text{and} \quad \langle Q_-\rangle^* @\{p\} = \langle p+\rangle W(-d(p)).$$

Here $\Delta$ is taken from the "counter-examples" section.

In other words,

$$Q_+ = \bigsqcup_{p\in\mathbb{R}} (@\{p\} \times^{\mathsf{FCD}} \langle p+\rangle W(d(p))); \quad Q_- = \bigsqcup_{p\in\mathbb{R}} (@\{p\} \times^{\mathsf{FCD}} \langle p+\rangle W(-d(p))).$$

That is $\langle Q_+\rangle^* @\{p\}$ and $\langle Q_-\rangle^* @\{p\}$ are something like infinitely small spherical sectors (with infinitely small aperture and infinitely small radius).

FiXme: Describe the co-complete funcoids reverse to these complete funcoids.

THEOREM 1901. A $D^1$ curve $f$ is an reparametrized integral curve for a direction field $d$ iff $f \in \mathrm{C}(\iota_D|\mathbb{R}|_>, Q_+) \cap \mathrm{C}(\iota_D|\mathbb{R}|_<, Q_-)$.

PROOF. Equivalently transform $f \in \mathrm{C}(\iota_D|\mathbb{R}|, Q_+)$; $f \circ \iota_D|\mathbb{R}| \sqsubseteq Q_+ \circ f$; $\langle f \circ \iota_D|\mathbb{R}|\rangle^* @\{t\} \sqsubseteq \langle Q_+ \circ f\rangle^* @\{t\}$; $\langle f\rangle^* \Delta_>(t) \sqcap D \sqsubseteq \langle Q_+\rangle^* f(t)$; $\langle f\rangle^* \Delta_>(t) \sqsubseteq \langle Q_+\rangle^* f(t)$; $\langle f\rangle^* \Delta_>(t) \sqsubseteq f(t) + W(D(f(t)))$; $\langle f\rangle^* \Delta_>(t) - f(t) \sqsubseteq W(D(f(t)))$;

$$\forall r>0, \phi>0 \exists \delta>0 : \langle f\rangle^* (]t; t+\delta[) - f(t) \subseteq R(d(f(t)),\phi) \cap B_r(f(t));$$

$$\forall r>0, \phi>0 \exists \delta>0 \forall 0<\gamma<\delta : \langle f\rangle^* (]t; t+\gamma[) - f(t) \subseteq R(d(f(t)),\phi) \cap B_r(f(t));$$

$$\forall r>0, \phi>0 \exists \delta>0 \forall 0<\gamma<\delta : \frac{\langle f\rangle^* (]t; t+\gamma[) - f(t)}{\gamma} \subseteq R(d(f(t)),\phi) \cap B_{r/\delta}(f(t));$$

$$\forall r>0, \phi>0 \exists \delta>0 : \partial_+ f(t) \subseteq R(d(f(t)),\phi) \cap B_{r/\delta}(f(t));$$

$$\forall r>0, \phi>0 : \partial_+ f(t) \subseteq R(d(f(t)),\phi);$$

$$\partial_+ f(t) \upuparrows d(f(t))$$

where $\partial_+$ is the right derivative.

In the same way we derive that $\mathrm{C}(|\mathbb{R}|_<, Q_-) \Leftrightarrow \partial_- f(t) \upuparrows d(f(t))$.

Thus $f'(t) \upuparrows d(f(t))$ iff $f \in \mathrm{C}(|\mathbb{R}|_>, Q_+) \cap \mathrm{C}(|\mathbb{R}|_<, Q_-)$. □

The following idea seems wrong. I grayed it out as a candidate for deletion from the text:

**6.4.** $C^n$ **curves.** FiXme: Related questions: http://math.stackexchange.com/q/1884856/4876 http://math.stackexchange.com/q/107460/4876 http://mathoverflow.net/q/88501

Define $R^n(d) = \left\{ \frac{v \in V}{\widehat{vd} < o(|v|^n), v \neq 0} \right\}$ for $d \neq 0$ and $R^n(0) = \{0\}$.

DEFINITION 1902. $W^n(d) = R^n(d) \sqcap \prod^{\mathsf{RLD}}_{r>0} B_r(0)$.

Finally our funcoids are the complete funcoids $Q^n_+$ and $Q^n_-$ described by the formulas

$$\langle Q^n_+ \rangle^* @\{p\} = \langle p+ \rangle W^n(d(p)) \quad \text{and} \quad \langle Q^n_- \rangle^* @\{p\} = \langle p+ \rangle W^n(-d(p)).$$

LEMMA 1903. Let for every $x$ in the domain of the path for small $t > 0$ we have $f(x+t) \in R^n(F(f(x)))$ and $f(x-t) \in R^n(-F(f(x)))$. Then $f$ is $C^n$ smooth.

PROOF. FiXme: Not yet proved!
See also http://math.stackexchange.com/q/1884930/4876. □

CONJECTURE 1904. A path $f$ is $C^n$ smooth iff $f \in \mathrm{C}(\iota_D |\mathbb{R}|_>, Q^n_+) \cap \mathrm{C}(\iota_D |\mathbb{R}|_<, Q^n_-)$.

PROOF. Reverse implication follows from the lemma.
Let now a path $f$ is $C^n$. Then

$$f(x+t) = \sum_{i=0}^{n} f^{(i)}(x)\frac{t^i}{i!} + o(t^i) = f(x) + f'(x)t + \sum_{i=2}^{n} f^{(i)}(x)\frac{t^i}{i!} + o(t^i)$$

□

**6.5. Plural funcoids.** Take $I_+$ and $Q_+$ as described above in forward direction and $I_-$ and $Q_-$ in backward direction. Then

$$f \in \mathrm{C}(I_+, Q_+) \wedge f \in \mathrm{C}(I_-, Q_-) \Leftrightarrow f \times f \in \mathrm{C}(I_+ \times^{(A)} I_-, Q_+ \times^{(A)} Q_-)?$$

To describe the above we can introduce new term *plural funcoids*. This is simply a map from an index set to funcoids. Composition is defined componentwise. Order is defined as product order. Well, do we need this? Isn't it the same as infimum product of funcoids?

**6.6. Multiple allowed directions per point.**

$$\langle Q \rangle^* @\{p\} = \bigsqcup_{d \in d(p)} \langle p+ \rangle W(d).$$

It seems (check!) that solutions not only of differential equations but also of difference equations can be expressed as paths in funcoids.

CHAPTER 3

# Generalized cofinite filters

The following is a straightforward generalization of cofinite filter on a coatomic poset.

DEFINITION 1905. $\Omega_{1a} = \bigsqcap^{\mathscr{F}}_{X \in \operatorname{coatoms}^{\mathfrak{A}}} X$; $\Omega_{1b} = \bigsqcap^{\mathscr{F}}_{X \in \operatorname{coatoms}^{\mathscr{F}}} X$.

PROPOSITION 1906. For primary filtrators $\Omega_{1a} = \Omega_{1b}$.

PROOF. Proposition 531. □

Thus for primary filtrators I will denote it just $\Omega$.

PROPOSITION 1907. Let $\mathfrak{A}$ be a subset of $\mathscr{P}U$. Let it be a meet-semilattice with greatest element FiXme: existence of greatest element seems unnecessary. Let also every non-coempty cofinite set lies in $\mathfrak{A}$. Then

$$\partial\Omega = \left\{ \frac{Y \in \mathfrak{A}}{\operatorname{card} \operatorname{atoms}^{\mathfrak{A}} Y \geq \omega} \right\}. \tag{1}$$

PROOF. $\Omega$ exists by corollary 496.

$Y \in \partial\Omega \Leftrightarrow Y \not\asymp^{\mathscr{F}} \bigsqcap^{\mathscr{F}}_{X \in \operatorname{coatoms}^{\mathfrak{A}}} X \Leftrightarrow$ (by properties of filter bases) $\Leftrightarrow \forall S \in \mathscr{P}_{\text{fin}} \operatorname{coatoms}^{\mathfrak{A}} : Y \not\asymp^{\mathscr{F}} \bigsqcap^{\mathscr{F}} S \Leftrightarrow$ (theorem 512) $\Leftrightarrow \forall S \in \mathscr{P}_{\text{fin}} \operatorname{coatoms}^{\mathfrak{A}} : Y \not\asymp \bigsqcap S \Leftrightarrow \forall K \in \mathscr{P}_{\text{fin}} U : Y \setminus K \neq \emptyset \Leftrightarrow \operatorname{card} Y \geq \omega \Leftrightarrow \operatorname{card} \operatorname{atoms}^{\mathfrak{A}} Y \geq \omega$. FiXme: Define $\mathscr{P}_{\text{fin}}$. □

COROLLARY 1908. Formula (1) holds for both reloids and funcoids.

PROOF. For reloiods it's straightforward, for funcoids take that they are isomorphic to filters on lattice $\Gamma$. □

COROLLARY 1909. $\Omega^{\mathsf{FCD}} \neq \bot^{\mathsf{FCD}}$ (for $\mathsf{FCD}(A, B)$ where $A \times B$ is an infinite set).

PROPOSITION 1910. $\operatorname{up} \Omega = \left\{ \frac{\bigsqcap S}{S \in \mathscr{P}_{\text{fin}} \operatorname{coatoms}^{\mathfrak{A}}} \right\}$.

PROOF. Because $\left\{ \frac{\bigsqcap S}{S \in \mathscr{P}_{\text{fin}} \operatorname{coatoms}^{\mathfrak{A}}} \right\}$ is a filter. □

COROLLARY 1911. $\operatorname{up} \Omega^{\mathsf{FCD}} = \operatorname{up} \Omega^{\mathsf{RLD}}$.

PROPOSITION 1912.

1°. $\langle \Omega^{\mathsf{FCD}} \rangle \{x\} = \Omega^U$;

2°. $\langle \Omega^{\mathsf{FCD}} \rangle p = \top$ for every nontrivial atomic filter $p$.

PROOF. $\langle \Omega^{\mathsf{FCD}} \rangle \{x\} = \bigsqcap^{\mathscr{F}}_{y \in U} (U \setminus \{y\}) = \Omega^U$; $\langle \Omega^{\mathsf{FCD}} \rangle p = \bigsqcap^{\mathscr{F}}_{y \in U} \top = \top$. □

PROPOSITION 1913. $(\mathsf{FCD})\Omega^{\mathsf{RLD}} = \Omega^{\mathsf{FCD}}$.

PROOF. $(\mathsf{FCD})\Omega^{\mathsf{RLD}} = \bigsqcap^{\mathsf{FCD}} \operatorname{up} \Omega^{\mathsf{RLD}} = \Omega^{\mathsf{FCD}}$. □

PROPOSITION 1914. $(\mathsf{RLD})_{\text{out}} \Omega^{\mathsf{FCD}} = \Omega^{\mathsf{RLD}}$.

PROOF. $(\mathsf{RLD})_{\text{out}} \Omega^{\mathsf{FCD}} = \bigsqcap^{\mathsf{RLD}} \operatorname{up} \Omega^{\mathsf{FCD}} = \bigsqcap^{\mathsf{RLD}} \operatorname{up} \Omega^{\mathsf{RLD}} = \Omega^{\mathsf{RLD}}$. □

PROPOSITION 1915. $(\mathsf{RLD})_{\mathrm{in}}\Omega^{\mathsf{FCD}} = \Omega^{\mathsf{RLD}}$.

PROOF.

$$
\begin{aligned}
(\mathsf{RLD})_{\mathrm{in}}\Omega^{\mathsf{FCD}} &= \bigsqcup\left\{\frac{a \times^{\mathsf{RLD}} b}{a \in \operatorname{atoms}^{\mathscr{F}}, b \in \operatorname{atoms}^{\mathscr{F}}, a \times^{\mathsf{FCD}} b \sqsubseteq \Omega^{\mathsf{FCD}}}\right\} = \\
&\bigsqcup\left\{\frac{a \times^{\mathsf{RLD}} b}{a \in \operatorname{atoms}^{\mathscr{F}}, b \in \operatorname{atoms}^{\mathscr{F}}, \text{not } a \text{ and } b \text{ both atomic}}\right\} = \\
&\bigsqcup\left\{\frac{\bigsqcup \operatorname{atoms}(a \times^{\mathsf{RLD}} b)}{a \in \operatorname{atoms}^{\mathscr{F}}, b \in \operatorname{atoms}^{\mathscr{F}}, \text{not } a \text{ and } b \text{ both atomic}}\right\} = \\
&\bigsqcup\bigcup\left\{\frac{\operatorname{atoms}(a \times^{\mathsf{RLD}} b)}{a \in \operatorname{atoms}^{\mathscr{F}}, b \in \operatorname{atoms}^{\mathscr{F}}, \text{not } a \text{ and } b \text{ both atomic}}\right\} = \\
&\bigsqcup(\text{nontrivial atomic reloids under } A \times B) = \Omega^{\mathsf{RLD}}.
\end{aligned}
$$

□

CHAPTER 4

# Extending Galois connections between funcoids and reloids

Definition 1916.

1°. $\Phi_* f = \lambda b \in \mathfrak{B} : \bigsqcup \left\{ \frac{x \in \mathfrak{A}}{f x \sqsubseteq b} \right\}$;

2°. $\Phi^* f = \lambda b \in \mathfrak{A} : \bigsqcap \left\{ \frac{x \in \mathfrak{B}}{f x \sqsupseteq b} \right\}$.

Proposition 1917.

1°. If $f$ has upper adjoint then $\Phi_* f$ is the upper adjoint of $f$.

2°. If $f$ has lower adjoint then $\Phi^* f$ is the lower adjoint of $f$.

Proof. By theorem 130. □

Lemma 1918. $\Phi^* (\mathsf{RLD})_{\mathrm{out}} = (\mathsf{FCD})$.

Proof. $(\Phi^* (\mathsf{RLD})_{\mathrm{out}}) f = \bigsqcap \left\{ \frac{g \in \mathsf{FCD}}{(\mathsf{RLD})_{\mathrm{out}} g \sqsupseteq f} \right\} = \bigsqcap^{\mathsf{FCD}} \left\{ \frac{g \in \mathbf{Rel}}{(\mathsf{RLD})_{\mathrm{out}} g \sqsupseteq f} \right\} = \bigsqcap^{\mathsf{FCD}} \left\{ \frac{g \in \mathbf{Rel}}{g \sqsupseteq f} \right\} = (\mathsf{FCD}) f$. □

Lemma 1919. $\Phi_* (\mathsf{RLD})_{\mathrm{out}} \neq (\mathsf{FCD})$.

Proof. $(\Phi_* (\mathsf{RLD})_{\mathrm{out}}) f = \bigsqcup \left\{ \frac{g \in \mathsf{FCD}}{(\mathsf{RLD})_{\mathrm{out}} g \sqsubseteq f} \right\}$
$(\Phi_* (\mathsf{RLD})_{\mathrm{out}}) \bot \neq \bot$. □

Lemma 1920. $\Phi^* (\mathsf{FCD}) = (\mathsf{RLD})_{\mathrm{out}}$.

Proof. $(\Phi^* (\mathsf{FCD})) f = \bigsqcap \left\{ \frac{g \in \mathsf{RLD}}{(\mathsf{FCD}) g \sqsupseteq f} \right\} = \bigsqcap^{\mathsf{RLD}} \left\{ \frac{g \in \mathbf{Rel}}{(\mathsf{FCD}) g \sqsupseteq f} \right\} = \bigsqcap^{\mathsf{RLD}} \left\{ \frac{g \in \mathbf{Rel}}{g \sqsupseteq f} \right\} = (\mathsf{RLD})_{\mathrm{out}} f$. □

Lemma 1921. $\Phi_* (\mathsf{RLD})_{\mathrm{in}} = (\mathsf{FCD})$.

Proof. $(\Phi_* (\mathsf{RLD})_{\mathrm{in}}) f = \bigsqcup \left\{ \frac{g \in \mathsf{FCD}}{(\mathsf{RLD})_{\mathrm{in}} g \sqsubseteq f} \right\} = \bigsqcup \left\{ \frac{g \in \mathsf{FCD}}{g \sqsubseteq (\mathsf{FCD}) f} \right\} = (\mathsf{FCD}) f$. □

Theorem 1922. The picture at figure 1 describes values of functions $\Phi_*$ and $\Phi^*$. All nodes of this diagram are distinct.

Proof. Follows from the above lemmas. □

Figure 1.

$$\begin{tikzcd}
(\mathsf{FCD}) \arrow[r, leftrightarrow, "\Phi_*"] \arrow[d, leftrightarrow, "\Phi^*"'] & (\mathsf{RLD})_{\mathrm{in}} \arrow[l, shift left=2, "\Phi^*"] \\
(\mathsf{RLD})_{\mathrm{out}} \arrow[d, "\Phi_*"] & \\
\text{other} &
\end{tikzcd}$$

QUESTION 1923. What happens if we keep applying $\Phi^*$ and $\Phi_*$ to the node "other"? Will we this way get a finite or infinite set?

CHAPTER 5

# Boolean funcoids

## 1. One-element boolean lattice

Let $\mathfrak{A}$ be a boolean lattice and $\mathfrak{B} = \mathscr{P}0$. It's sole element is $\bot$.

$f \in \mathsf{pFCD}(\mathfrak{A};\mathfrak{B}) \Leftrightarrow \forall X \in \mathfrak{A} : (\langle f\rangle X \not\asymp \bot \Leftrightarrow \langle f^{-1}\rangle\bot \not\asymp X) \Leftrightarrow \forall X \in \mathfrak{A} : (0 \Leftrightarrow \langle f^{-1}\rangle\bot \not\asymp X) \Leftrightarrow \forall X \in \mathfrak{A} : \langle f^{-1}\rangle\bot \asymp X \Leftrightarrow \forall X \in \mathfrak{A} : \langle f^{-1}\rangle\bot = \bot^{\mathfrak{A}} \Leftrightarrow \langle f^{-1}\rangle\bot = \bot^{\mathfrak{A}} \Leftrightarrow \langle f^{-1}\rangle = \{(\bot;\bot^{\mathfrak{A}})\}$.

Thus $\operatorname{card}\mathsf{pFCD}(\mathfrak{A};\mathscr{P}0) = 1$.

## 2. Two-element boolean lattice

Consider the two-element boolean lattice $\mathfrak{B} = \mathscr{P}1$.

Let $f$ be a pointfree protofuncoid from $\mathfrak{A}$ to $\mathfrak{B}$ (that is $(\mathfrak{A};\mathfrak{B};\alpha;\beta)$ where $\alpha \in \mathfrak{B}^{\mathfrak{A}}$, $\beta \in \mathfrak{A}^{\mathfrak{B}}$).

$f \in \mathsf{pFCD}(\mathfrak{A};\mathfrak{B}) \Leftrightarrow \forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\langle f\rangle X \not\asymp Y \Leftrightarrow \langle f^{-1}\rangle Y \not\asymp X) \Leftrightarrow \forall X \in \mathfrak{A}, Y \in \mathfrak{B} : ((0 \in \langle f\rangle X \wedge 0 \in Y) \vee (1 \in \langle f\rangle X \wedge 1 \in Y) \Leftrightarrow \langle f^{-1}\rangle Y \not\asymp X)$.

$T = \left\{ \frac{X \in \mathfrak{A}}{0 \in \langle f\rangle X} \right\}$ is an ideal. Really: That it's an upper set is obvious. Let $P \cup Q \in \left\{ \frac{X \in \mathfrak{A}}{0 \in \langle f\rangle X} \right\}$. Then $0 \in \langle f\rangle(P \cup Q) = \langle f\rangle P \cup \langle f\rangle Q$; $0 \in \langle f\rangle P \vee 0 \in \langle f\rangle Q$.

Similarly $S = \left\{ \frac{X \in \mathfrak{A}}{1 \in \langle f\rangle X} \right\}$ is an ideal.

Let now $T, S \in \mathscr{P}\mathfrak{A}$ be ideals. Can we restore $\langle f\rangle$? Yes, because we know $0 \in \langle f\rangle X$ and $1 \in \langle f\rangle X$ for every $X \in \mathfrak{A}$.

So it is equivalent to $\forall X \in \mathfrak{A}, Y \in \mathfrak{B} : ((X \in T \wedge 0 \in Y) \vee (X \in S \wedge 1 \in Y) \Leftrightarrow \langle f^{-1}\rangle Y \not\asymp X)$.

$f \in \mathsf{pFCD}(\mathfrak{A};\mathfrak{B})$ is equivalent to conjunction of all rows of this table:

| $Y$ | equality |
|---|---|
| $\emptyset$ | $\langle f^{-1}\rangle\emptyset = \emptyset$ |
| $\{0\}$ | $X \in T \Leftrightarrow \langle f^{-1}\rangle\{0\} \not\asymp X$ |
| $\{1\}$ | $X \in S \Leftrightarrow \langle f^{-1}\rangle\{1\} \not\asymp X$ |
| $\{0{,}1\}$ | $X \in T \vee X \in S \Leftrightarrow \langle f^{-1}\rangle\{0,1\} \not\asymp X$ |

Simplified:

| $Y$ | equality |
|---|---|
| $\emptyset$ | $\langle f^{-1}\rangle\emptyset = \emptyset$ |
| $\{0\}$ | $T = \partial\langle f^{-1}\rangle\{0\}$ |
| $\{1\}$ | $S = \partial\langle f^{-1}\rangle\{1\}$ |
| $\{0{,}1\}$ | $T \cup S = \partial\langle f^{-1}\rangle\{0,1\}$ |

From the last table it follows that $T$ and $S$ are principal ideals.

So we can take arbitrary either $\langle f^{-1}\rangle\{0\}$, $\langle f^{-1}\rangle\{1\}$ or principal ideals $T$ and $S$.

In other words, we take $\langle f^{-1}\rangle\{0\}$, $\langle f^{-1}\rangle\{1\}$ arbitrary and independently. So we have $\mathsf{pFCD}(\mathfrak{A};\mathfrak{B})$ equivalent to product of two instances of $\mathfrak{A}$. So it a boolean lattice. FiXme: I messed product with disjoint union below.)

## 3. Finite boolean lattices

We can assume $\mathfrak{B} = \mathscr{P}B$ for a set $B$, $\operatorname{card} B = n$. Then

$f \in \mathsf{pFCD}(\mathfrak{A}; \mathfrak{B}) \Leftrightarrow \forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\langle f \rangle X \not\asymp Y \Leftrightarrow \langle f^{-1} \rangle Y \not\asymp X) \Leftrightarrow \forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\exists i \in Y : i \in \langle f \rangle X \Leftrightarrow \langle f^{-1} \rangle Y \not\asymp X)$.

Having values of $\langle f^{-1} \rangle \{i\}$ we can restore all $\langle f^{-1} \rangle Y$. [need this paragraph?]

Let $T_i = \left\{ \frac{X \in \mathfrak{A}}{i \in \langle f \rangle X} \right\}$.

Let now $T_i \in \mathscr{P}\mathfrak{A}$ be ideals. Can we restore $\langle f \rangle$? Yes, because we know $i \in \langle f \rangle X$ for every $X \in \mathfrak{A}$.

So, it is equivalent to:

$$\forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\exists i \in Y : X \in T_i \Leftrightarrow \langle f^{-1} \rangle Y \not\asymp X). \tag{2}$$

Lemma 1924. The formula (2) is equivalent to:

$$\forall X \in \mathfrak{A}, i \in B : (X \in T_i \Leftrightarrow \langle f^{-1} \rangle \{i\} \not\asymp X). \tag{3}$$

Proof. (2)⇒(3). Just take $Y = \{i\}$.

(3)⇒(2). Let (3) holds. Let also $X \in \mathfrak{A}, Y \in \mathfrak{B}$. Then $\langle f^{-1} \rangle Y \not\asymp X \Leftrightarrow \bigcup_{i \in Y} \langle f^{-1} \rangle \{i\} \not\asymp X \Leftrightarrow \exists i \in Y : \langle f^{-1} \rangle \{i\} \not\asymp X \Leftrightarrow \exists i \in Y : X \in T_i$.

□

Further transforming: $\forall i \in B : T_i = \partial \langle f^{-1} \rangle \{i\}$.

So $\langle f^{-1} \rangle \{i\}$ are arbitary elements of $\mathfrak{B}$ and $T_i$ are corresponding arbitrary principal ideals.

In other words, $\mathsf{pFCD}(\mathfrak{A}; \mathfrak{B}) \cong \mathfrak{A} \Pi \ldots \Pi \mathfrak{A}$ ($\operatorname{card} B$ times). Thus $\mathsf{pFCD}(\mathfrak{A}; \mathfrak{B})$ is a boolean lattice.

## 4. About infinite case

Let $\mathfrak{A}$ be a complete boolean lattice, $\mathfrak{B}$ be an atomistic boolean lattice.

$f \in \mathsf{pFCD}(\mathfrak{A}; \mathfrak{B}) \Leftrightarrow \forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\langle f \rangle X \not\asymp Y \Leftrightarrow \langle f^{-1} \rangle Y \not\asymp X) \Leftrightarrow \forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\exists i \in \operatorname{atoms} Y : i \in \operatorname{atoms} \langle f \rangle X \Leftrightarrow \langle f^{-1} \rangle Y \not\asymp X)$.

Let $T_i = \left\{ \frac{X \in \mathfrak{A}}{i \in \operatorname{atoms} \langle f \rangle X} \right\}$.

$T_i$ is an ideal: Really: That it's an upper set is obvious. Let $P \cup Q \in \left\{ \frac{X \in \mathfrak{A}}{i \in \operatorname{atoms} \langle f \rangle X} \right\}$. Then $i \in \operatorname{atoms} \langle f \rangle (P \cup Q) = \operatorname{atoms} \langle f \rangle P \cup \operatorname{atoms} \langle f \rangle Q$; $i \in \langle f \rangle P \vee i \in \langle f \rangle Q$.

Let now $T_i \in \mathscr{P}\mathfrak{A}$ be ideals. Can we restore $\langle f \rangle$? Yes, because we know $i \in \operatorname{atoms} \langle f \rangle X$ for every $X \in \mathfrak{A}$ and $\mathfrak{B}$ is atomistic.

So, it is equivalent to:

$$\forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\exists i \in \operatorname{atoms} Y : X \in T_i \Leftrightarrow \langle f^{-1} \rangle Y \not\asymp X). \tag{4}$$

Lemma 1925. The formula (4) is equivalent to:

$$\forall X \in \mathfrak{A}, i \in \operatorname{atoms}^{\mathfrak{B}} : (X \in T_i \Leftrightarrow \langle f^{-1} \rangle i \not\asymp X). \tag{5}$$

Proof. (4)⇒(5). Let (4) holds. Take $Y = i$. Then $\operatorname{atoms} Y = \{i\}$ and thus $X \in T_i \Leftrightarrow \exists i \in \operatorname{atoms} Y : X \in T_i \Leftrightarrow \langle f^{-1} \rangle Y \not\asymp X \Leftrightarrow \langle f^{-1} \rangle i \not\asymp X$.

(5)⇒(4). Let (3) holds. Let also $X \in \mathfrak{A}$, $Y \in \mathfrak{B}$. Then $\langle f^{-1} \rangle Y \not\asymp X \Leftrightarrow \langle f^{-1} \rangle \bigsqcup \operatorname{atoms} Y \not\asymp X \Leftrightarrow \bigsqcup_{i \in \operatorname{atoms} Y} \langle f^{-1} \rangle i \not\asymp X \Leftrightarrow \exists i \in \operatorname{atoms} Y : \langle f^{-1} \rangle i \not\asymp X \Leftrightarrow \exists i \in \operatorname{atoms} Y : X \in T_i$.

□

Further equivalently transforming: $\forall i \in \operatorname{atoms}^{\mathfrak{B}} : T_i = \partial \langle f^{-1} \rangle i$.

So $\langle f^{-1} \rangle i$ are arbitary elements of $\mathfrak{B}$ and $T_i$ are corresponding arbitrary principal ideals.

In other words, $\mathsf{pFCD}(\mathfrak{A};\mathfrak{B}) \cong \prod_{i\in\operatorname{card}\operatorname{atoms}^{\mathfrak{B}}} \mathfrak{A}$. Thus $\mathsf{pFCD}(\mathfrak{A};\mathfrak{B})$ is a boolean lattice.

So finally we have a very weird theorem, which is a partial solution for the above open problem (The weirdness is in its partiality and asymmetry):

THEOREM 1926. If $\mathfrak{A}$ is a complete boolean lattice and $\mathfrak{B}$ is an atomistic boolean lattice (or vice versa), then $\mathsf{pFCD}(\mathfrak{A};\mathfrak{B})$ is a boolean lattice.

[4] proves "THEOREM 4.6. Let $A$, $B$ be bounded posets. $A \otimes B$ is a completely distributive complete Boolean lattice iff $A$ and $B$ are completely distributive Boolean lattices." (where $A \otimes B$ is equivalent to the set of Galois connections between $A$ and $B$) and other interesting results.

CHAPTER 6

# Interior funcoids

Having a funcoid $f$ let define *interior funcoid* $f^\circ$.

DEFINITION 1927. Let $f \in \mathsf{FCD}(A, B) = \mathsf{pFCD}(\mathscr{T} A, \mathscr{T} B)$ be a co-complete funcoid. Then $f^\circ \in \mathsf{pFCD}(\operatorname{dual} \mathscr{T} A, \operatorname{dual} \mathscr{T} B)$ is defined by the formula $\langle f^\circ \rangle^* X = \overline{\langle f \rangle \overline{X}}$.

PROPOSITION 1928. Pointfree funcoid $f^\circ$ exists and is unique.

PROOF. $X \mapsto \overline{\langle f \rangle \overline{X}}$ is a component of pointfree funcoid $\operatorname{dual} \mathscr{T} A \to \operatorname{dual} \mathscr{T} B$ iff $\langle f \rangle$ is a component of the corresponding pointfree funcoid $\mathscr{T} A \to \mathscr{T} B$ that is essentially component of the corresponding funcoid $\mathsf{FCD}(A, B)$ what holds for a unique funcoid. □

It can be also defined for arbitrary funcoids by the formula $f^\circ = (\operatorname{CoCompl} f)^\circ$.

OBVIOUS 1929. $f^\circ$ is co-complete.

THEOREM 1930. The following values are pairwise equal for a co-complete funcoid $f$ and $X \in \mathscr{T} \operatorname{Src} f$:

1°. $\langle f^\circ \rangle^* X$;
2°. $\left\{ \frac{y \in \operatorname{Dst} f}{\langle f^{-1} \rangle^* \{y\} \sqsubseteq X} \right\}$
3°. $\bigsqcup \left\{ \frac{Y \in \mathscr{T} \operatorname{Dst} f}{\langle f^{-1} \rangle^* Y \sqsubseteq X} \right\}$
4°. $\bigsqcup \left\{ \frac{\mathcal{Y} \in \mathscr{F} \operatorname{Dst} f}{\langle f^{-1} \rangle \mathcal{Y} \sqsubseteq X} \right\}$

PROOF.

1°=2°. $\left\{ \frac{y \in \operatorname{Dst} f}{\langle f^{-1} \rangle^* \{y\} \sqsubseteq X} \right\} = \left\{ \frac{x \in \operatorname{Dst} f}{\langle f^{-1} \rangle^* \{x\} \asymp \overline{X}} \right\} = \left\{ \frac{x \in \operatorname{Dst} f}{\{x\} \asymp \langle f \rangle \overline{X}} \right\} = \overline{\langle f \rangle \overline{X}} = \langle f^\circ \rangle^* X$.

2°=3°. If $\left\langle f^{-1} \right\rangle^* Y \sqsubseteq X$ then (by completeness of $f^{-1}$) $Y = \left\{ \frac{y \in Y}{\langle f^{-1} \rangle^* \{y\} \sqsubseteq X} \right\}$ and thus
$$\bigsqcup \left\{ \frac{Y \in \mathscr{T} \operatorname{Dst} f}{\langle f^{-1} \rangle^* Y \sqsubseteq X} \right\} \sqsubseteq \left\{ \frac{y \in \operatorname{Dst} f}{\langle f^{-1} \rangle^* \{y\} \sqsubseteq X} \right\}.$$
The reverse inequality is obvious.

3°=4°. It's enough to prove that if $\left\langle f^{-1} \right\rangle \mathcal{Y} \sqsubseteq X$ for $\mathcal{Y} \in \mathscr{F} \operatorname{Dst} f$ then exists $Y \in \operatorname{up} \mathcal{Y}$ such that $\langle f^{-1} \rangle^* Y \sqsubseteq X$. Really let $\left\langle f^{-1} \right\rangle \mathcal{Y} \sqsubseteq X$. Then $\bigsqcap \langle \langle f^{-1} \rangle^* \rangle^* \operatorname{up} \mathcal{Y} \sqsubseteq X$ and thus exists $Y \in \operatorname{up} \mathcal{Y}$ such that $\langle f^{-1} \rangle^* Y \sqsubseteq X$ by properties of generalized filter bases.

□

This coincides with the customary definition of interior in topological spaces.

PROPOSITION 1931. $f^{\circ\circ} = f$ for every funcoid $f$.

PROOF. $\langle f^{\circ\circ} \rangle^* X = \neg\neg \langle f \rangle \neg\neg X = \langle f \rangle X$. □

PROPOSITION 1932. Let $g \in \mathsf{FCD}(A, B)$, $f \in \mathsf{FCD}(B, C)$, $h \in \mathsf{FCD}(A, C)$ for some sets $A$. $B$, $C$.

$g \sqsubseteq f^\circ \circ h \Leftrightarrow f^{-1} \circ g \sqsubseteq h$, provided $f$ and $h$ are co-complete.

PROOF. $g \sqsubseteq f^\circ \circ h \Leftrightarrow \forall X \in A : \langle g \rangle^* X \sqsubseteq \langle f^\circ \circ h \rangle^* X \Leftrightarrow \forall X \in A : \langle g \rangle^* X \sqsubseteq \langle f^\circ \rangle^* \langle h \rangle^* X \Leftrightarrow \forall X \in A : \langle g \rangle^* X \sqsubseteq \neg \langle f \rangle^* \neg \langle h \rangle^* X \Leftrightarrow \forall X \in A : \langle g \rangle^* X \asymp \langle f \rangle^* \neg \langle h \rangle^* X \Leftrightarrow \forall X \in A : \left\langle f^{-1} \right\rangle^* \langle g \rangle^* X \asymp \neg \langle h \rangle^* X \Leftrightarrow \forall X \in A : \left\langle f^{-1} \right\rangle^* \langle g \rangle^* X \sqsubseteq \langle h \rangle^* X \Leftrightarrow \forall X \in A : \left\langle f^{-1} \circ g \right\rangle^* X \sqsubseteq \langle h \rangle^* X \Leftrightarrow f^{-1} \circ g \sqsubseteq h$. □

REMARK 1933. The above theorem allows to get rid of interior funcoids (and use only "regular" funcoids) in some formulas.

CHAPTER 7

# Filterization of pointfree funcoids

Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. By corollary 502 we have that $\mathfrak{A}$ and $\mathfrak{B}$ are complete lattices.

Let $f$ be a pointfree funcoid $\mathfrak{Z}_0 \to \mathfrak{Z}_1$. Define pointfree funcoid $\uparrow f$ (*filterization* of $f$) by the formulas

$$\langle \uparrow f \rangle \mathcal{X} = \bigsqcap^{\mathfrak{B}}_{X \in \operatorname{up} \mathcal{X}} \langle f \rangle X \quad \text{and} \quad \left\langle \uparrow f^{-1} \right\rangle \mathcal{Y} = \bigsqcap^{\mathfrak{A}}_{Y \in \operatorname{up} \mathcal{Y}} \left\langle f^{-1} \right\rangle Y.$$

Proposition 1934. $\uparrow f$ is a pointfree funcoid.

Proof.

$$\begin{aligned}
\mathcal{Y} \not\asymp \langle \uparrow f \rangle \mathcal{X} \Leftrightarrow \mathcal{Y} \not\asymp \bigsqcap^{\mathfrak{B}}_{X \in \operatorname{up} \mathcal{X}} \langle f \rangle X \Leftrightarrow &\\
\bigsqcap^{\mathfrak{B}}_{X \in \operatorname{up} \mathcal{X}} (\mathcal{Y} \sqcap^{\mathfrak{B}} \langle f \rangle X) \neq \bot \Leftrightarrow &\text{ (corollary 552*)}\\
\forall X \in \operatorname{up} \mathcal{X} : \mathcal{Y} \sqcap^{\mathfrak{B}} \langle f \rangle X \neq \bot \Leftrightarrow &\text{ (theorem 520)}\\
\forall X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y} : Y \sqcap^{\mathfrak{B}} \langle f \rangle X \neq \bot \Leftrightarrow &\text{ (corollary 519)}\\
\forall X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y} : Y \sqcap^{\mathfrak{Z}_1} \langle f \rangle X \neq \bot \Leftrightarrow &\\
\forall X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y} : X \, [f] \, Y.&
\end{aligned}$$

* To apply corollary 552 we need to show that $\left\{ \frac{\mathcal{Y} \sqcap^{\mathfrak{B}} \langle f \rangle X}{X \in \operatorname{up} \mathcal{X}} \right\}$ is a generalized filter base. To show it is enough to show that $\left\{ \frac{\langle f \rangle X}{X \in \operatorname{up} \mathcal{X}} \right\}$ is a generalized filter base. But this easily follows from proposition 1401 and 558.

Similarly $\mathcal{X} \not\asymp \left\langle \uparrow f^{-1} \right\rangle \mathcal{Y} \Leftrightarrow \forall X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y} : X \, [f] \, Y$. Thus $\mathcal{Y} \not\asymp \langle \uparrow f \rangle \mathcal{X} \Leftrightarrow \mathcal{X} \not\asymp \left\langle \uparrow f^{-1} \right\rangle \mathcal{Y}$. □

Proposition 1935. The above defined $\uparrow$ is an injection.

Proof. $\langle \uparrow f \rangle X = \bigsqcap^{\mathfrak{B}}_{X' \in \operatorname{up} X} \langle f \rangle X' = \min_{X' \in \operatorname{up} X} \langle f \rangle X' = \langle f \rangle X$. So $\langle f \rangle$ is determined by $\langle \uparrow f \rangle$. Likewise $\left\langle f^{-1} \right\rangle$ is determined by $\left\langle \uparrow f^{-1} \right\rangle$. □

Conjecture 1936. (Non generalizing of theorem 1510) Pointfree funcoids $f$ between some: a. atomistic but non-complete; b. complete but non-atomistic boolean lattices $\mathfrak{Z}_0$ and $\mathfrak{Z}_1$ do not bijectively correspond to morphisms $p \in$ **Rel**$(\operatorname{atoms} \mathfrak{Z}_0, \operatorname{atoms} \mathfrak{Z}_1)$ by the formulas:

$$\langle f \rangle X = \bigsqcup \langle p \rangle^* \operatorname{atoms} X, \quad \left\langle f^{-1} \right\rangle Y = \bigsqcup \left\langle p^{-1} \right\rangle^* \operatorname{atoms} Y;$$

$$(x, y) \in \operatorname{GR} p \Leftrightarrow y \in \operatorname{atoms} \langle f \rangle x \Leftrightarrow x \in \operatorname{atoms} \left\langle f^{-1} \right\rangle y.$$

CHAPTER 8

# Systems of sides

Now we will consider a common generalization of (some of pointfree) funcoids and (some of) Galois connections. The main purpose of this is general theorem 1984 below.

First consider some properties of Galois connections:

## 1. More on Galois connections

Here I will denote $\langle f\rangle$ the lower adjoint of a Galois connection $f$. FiXme: Switch to this notation in the book?

Let $\mathsf{GAL}$ be the category of Galois connections. FiXme: Need to decide whether use $\mathsf{GAL}(A,B)$ or $A\otimes B$.

I will denote $(f,g)^{-1}=(g,f)$ for a Galois connection $(f,g)$.

We will order Galois connections by the formula

$$f\sqsubseteq g\Leftrightarrow\langle f\rangle\sqsubseteq\langle g\rangle\Leftrightarrow\left\langle f^{-1}\right\rangle\sqsupseteq\left\langle g^{-1}\right\rangle.$$

Obvious 1937. This defines a partial order on the set of Galois connections between any two (fixed) posets.

Proposition 1938. If $f$ and $g$ are Galois connections (between a join-semilattice $\mathfrak{A}$ and a meet-semilattice $\mathfrak{B}$), then there exists a Galois connection $f\sqcup g$ determined by the formula $\langle f\sqcup g\rangle x=\langle f\rangle x\sqcup\langle g\rangle x$.

Proof. It is enough to prove that

$$(x\mapsto\langle f\rangle x\sqcup\langle g\rangle x,y\mapsto\left\langle f^{-1}\right\rangle y\sqcap\left\langle g^{-1}\right\rangle y)$$

is a Galois connection that is that

$$\langle f\rangle x\sqcup\langle g\rangle x\sqsubseteq y\Leftrightarrow x\sqsubseteq\left\langle f^{-1}\right\rangle y\sqcap\left\langle g^{-1}\right\rangle y$$

for all relevant $x$ and $y$.

Really,

$$\langle f\rangle x\sqcup\langle g\rangle x\sqsubseteq y\Leftrightarrow\langle f\rangle x\sqsubseteq y\wedge\langle g\rangle x\sqsubseteq y\Leftrightarrow\\ x\sqsubseteq\left\langle f^{-1}\right\rangle y\wedge x\sqsubseteq\left\langle g^{-1}\right\rangle y\Leftrightarrow x\sqsubseteq\left\langle f^{-1}\right\rangle y\sqcap\left\langle g^{-1}\right\rangle y.$$

□

FiXme: Describe infinite join of Galois connections.

Proposition 1939. If $\mathfrak{A}$ is a poset with least element, then $\langle a\rangle\bot=\bot$.

Proof. $\langle a\rangle\bot\sqsubseteq y\Leftrightarrow\bot\sqsubseteq\left\langle a^{-1}\right\rangle y\Leftrightarrow 1$. Thus $\langle a\rangle\bot$ is the least element. □

Proposition 1940. $(\mathfrak{A}\times\{\bot^{\mathfrak{B}}\},\mathfrak{B}\times\{\top^{\mathfrak{A}}\})$ is the least Galois connection from a poset $\mathfrak{A}$ with greatest element to a poset $\mathfrak{B}$ with least element.

Proof. Let's prove that it is a Galois connection. We need to prove

$$(\mathfrak{A}\times\{\bot^{\mathfrak{B}}\})x\sqsubseteq y\Leftrightarrow x\sqsubseteq(\mathfrak{B}\times\{\top^{\mathfrak{A}}\})y.$$

But this is trivially equivalent to $1\Leftrightarrow 1$. Thus it's a Galois connection.

That it the least is obvious. □

COROLLARY 1941. $\langle\bot\rangle x = \bot$ for Galois connections from a poset $\mathfrak{A}$ with greatest element to a poset $\mathfrak{B}$ with least element. FiXme: Clarify.

THEOREM 1942. If $\mathfrak{A}$ and $\mathfrak{B}$ are bounded posets, then $\mathsf{GAL}(\mathfrak{A},\mathfrak{B})$ is bounded.

PROOF. That $\mathsf{GAL}(\mathfrak{A},\mathfrak{B})$ has least element was proved above. I will demonstrate that $(\alpha,\beta)$ is the greatest element of $\mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$ for

$$\alpha X = \begin{cases} \bot^{\mathfrak{B}} & \text{if } X = \bot^{\mathfrak{A}} \\ \top^{\mathfrak{B}} & \text{if } X \neq \bot^{\mathfrak{A}} \end{cases}; \quad \beta Y = \begin{cases} \top^{\mathfrak{A}} & \text{if } Y = \top^{\mathfrak{B}} \\ \bot^{\mathfrak{A}} & \text{if } Y \neq \top^{\mathfrak{B}} \end{cases}.$$

First prove $Y \sqsubseteq \alpha X \Leftrightarrow X \sqsubseteq \beta Y$.

Really $\alpha X \sqsubseteq Y \Leftrightarrow X = \bot^{\mathfrak{A}} \vee Y = \top^{\mathfrak{B}} \Leftrightarrow X \sqsubseteq \beta Y$.

That it is the greatest Galois connection between $\mathfrak{A}$ and $\mathfrak{B}$ easily follows from proposition 1939. □

THEOREM 1943. For every brouwerian lattice $x \mapsto c \sqcap x$ is a lower adjoint.

PROOF. By dual of theorem 153. □

EXERCISE 1944. Describe the corresponding upper adjoint, especially for the special case of boolean lattices.

## 2. Definition

DEFINITION 1945. *System of presides* is a functor $\Upsilon = (f \mapsto \langle f\rangle)$ from an ordered category to the category of functions between (small) bounded lattices, such that (for all relevant variables):

1°. Every Hom-set of $\operatorname{Src}\Upsilon$ is a bounded join-semilattice.
2°. $\langle a\rangle\bot = \bot$.
3°. $\langle a \sqcup b\rangle X = \langle a\rangle X \sqcup \langle b\rangle X$ (equivalent to $\Upsilon$ to be a join-semilattice homomorphism, if we order functions between small bounded lattices componentwise).

I call morphisms of such categories *sides.*[1]

REMARK 1946. We could generalize to functions between small join-semilattices with least elements instead of bounded lattices only, but this is not really necessary.

DEFINITION 1947. I will call objects of the source category of this functor simply *objects of the presides.*

DEFINITION 1948. *Bounded* system of presides is system of presides from an ordered category with bounded Hom-sets such that $X, Y \in \operatorname{Ob}\operatorname{Src}\Upsilon$ the following additional axioms hold for all suitable $a$:

1°. $\left\langle \bot^{\operatorname{Hom}(X,Y)}\right\rangle a = \bot$.
2°. $\left\langle \top^{\operatorname{Hom}(X,Y)}\right\rangle a = \top$ unless $a = \bot$

DEFINITION 1949. *System of presides with identities* is a system of presides with a morphism $\mathrm{id}_a \in \operatorname{Src}\Upsilon$ for every object $\mathfrak{A}$ of $\operatorname{Src}\Upsilon$ and $a \in \mathfrak{A}$ and the following additional axioms:

1°. $\mathrm{id}_c \sqsubseteq 1_{\mathfrak{A}}$ for every $c \in \mathfrak{A}$ where $\mathfrak{A}$ is an object of $\operatorname{Src}\Upsilon$.
2°. $\langle \mathrm{id}_c\rangle = (\lambda x \in \mathfrak{A} : x \sqcap c)$ for every $c \in \mathfrak{A}$ where $\mathfrak{A}$ is an object of $\operatorname{Src}\Upsilon$

DEFINITION 1950. *System of sides* is a system of presides which is both bounded and with identities.

[1]The idea for the name is that we consider one "side" $\langle f\rangle$ of a funcoid instead of both sides $\langle f\rangle$ and $\left\langle f^{-1}\right\rangle$.

Proposition 1951. $\langle 1_{\mathfrak{A}}^{\operatorname{Src}\Upsilon}\rangle a = a$ for every system of presides.

Proof. By properties of functors. □

Definition 1952. I call a system of *monotone* presides a system of presides with additional axiom:

1°. $\langle a\rangle$ is monotone.

Definition 1953. I call a system of *distributive* presides a system of presides with additional axiom:

1°. $\langle a\rangle(X \sqcup Y) = \langle a\rangle X \sqcup \langle a\rangle Y$.

Obvious 1954. Every distributive system of presides is monotone.

Proposition 1955. $\langle a \sqcap b\rangle X \sqsubseteq \langle a\rangle X \sqcap \langle b\rangle X$ for monotone systems of sides if Hom-sets are lattices.

Definition 1956. A system of presides *with correct identities* is a system of presides with identities with additional axiom:

1°. $\mathrm{id}_b \circ \mathrm{id}_a = \mathrm{id}_{a\sqcap b}$.

Proposition 1957. Every faithful system of presides with identities is with correct identities.

Proof. $\langle \mathrm{id}_b \circ \mathrm{id}_a\rangle x = (\langle \mathrm{id}_b\rangle \circ \langle \mathrm{id}_a\rangle)x = \langle \mathrm{id}_b\rangle\langle \mathrm{id}_a\rangle x = b \sqcap a \sqcap x = \langle \mathrm{id}_{b\sqcap a}\rangle x$. Thus by faithfulness $\mathrm{id}_b \circ \mathrm{id}_a = \mathrm{id}_{b\sqcap a} = \mathrm{id}_{a\sqcap b}$. □

Definition 1958. *Restricting* a side $f$ to an object $X$ is defined by the formula $f|_X = f \circ \mathrm{id}_X$.

Definition 1959. *Image* of a preside is defined by the formula $\operatorname{im} f = \langle f\rangle\top$.

Definition 1960. Protofuncoids *over* a set $X$ of functors is a protofuncoid $f$ such that $\langle f\rangle \in X \wedge \langle f^{-1}\rangle \in X$.

## 3. Concrete examples of sides

Obvious 1961. The category **Rel** with $\langle f\rangle = \langle f\rangle^*$ for $f \in \mathbf{Rel}$ and usual $\mathrm{id}_c$ defines a distributive system of sides with correct identities.

### 3.1. Some subsides.

Definition 1962. *Full subsystem* of a system $\Upsilon$ of presides is the functor $\Upsilon$ restricted to a full subcategory of $\operatorname{Src}\Upsilon$.

Obvious 1963. Full subsystem of a system of presides is always a system of presides.

Obvious 1964. Full subsystem of a bounded system of presides is always a bounded subsystem of presides.

Obvious 1965.

1°. Full subsystem of a system of presides with identities is always with identities.
2°. Full subsystem of a system of presides with correct identities is always with correct identities.

Obvious 1966. Full subsystem of a distributive system of presides is always a distributive system of presides.

Obvious 1967. Full subsystem of a system of sides is always a system of sides.

### 3.2. Funcoids and pointfree funcoids.

PROPOSITION 1968. The category of pointfree funcoids between starrish join-semilattices with usual $\langle f\rangle$ defines a system of presides.

PROOF. Theorem 1430. □

PROPOSITION 1969. The category of pointfree funcoids between bounded starrish join-semilattices with usual $\langle f\rangle$ defines a system of bounded presides.

PROOF. Take the proof of theorem 1427 into account. □

PROPOSITION 1970. The category of pointfree funcoids from a starrish join-semilattices to a separable starrish join-semilattices defines a distributive system of presides.

PROOF. Theorem 1402. □

PROPOSITION 1971. The category of pointfree funcoids between starrish lattices with usual $\langle f\rangle$ and usual $\mathrm{id}_c$ defines a system of presides with correct identities.

PROOF. That it is with identities is obvious.
That it is with correct identities is obvious. □

OBVIOUS 1972. The category of pointfree funcoids between bounded starrish lattices with usual $\langle f\rangle$ and usual $\mathrm{id}_c$ defines a system of sides with correct identities.

PROPOSITION 1973. The category of funcoids with usual $\langle f\rangle$ and usual $\mathrm{id}_c$ defines a system of sides with correct identities.

PROOF. Because it can be considered a full subsystem of the category of pointfree funcoids between bounded starrish lattices with usual $\langle f\rangle$. □

### 3.3. Galois connections.

PROPOSITION 1974. The category of Galois connections between (small) lattices with least elements together with usual $\langle f\rangle$ defines a distributive system of presides.

PROOF. Propositions 1938 and 1939 for a system of presides.
It is distributive because lower adjoints preserve all joins. □

PROPOSITION 1975. The category of Galois connections between (small) bounded lattices together with usual $\langle f\rangle$ defines a bounded system of presides.

PROOF. Theorem 1942. □

PROPOSITION 1976. The category of Galois connections between (small) Heyting lattices together with usual $\langle f\rangle$ defines a system of sides with correct identities.

PROOF. Theorem 1943 ensures that they a system of sides with identities. The identities are correct due to faithfulness. □

### 3.4. Reloids.

PROPOSITION 1977. Reloids with the functor $f\mapsto\langle(\mathsf{FCD})f\rangle$ and usual $\mathrm{id}_c$ form a system of sides with correct identities.

PROOF. It is really a functor because $\langle(\mathsf{FCD})g\rangle\circ\langle(\mathsf{FCD})f\rangle = \langle(\mathsf{FCD})g\circ(\mathsf{FCD})f\rangle = \langle(\mathsf{FCD})(g\circ f)\rangle$ for every composable reloids $f$ and $g$.

$\langle a\rangle\bot = \langle(\mathsf{FCD})a\rangle\bot = \bot$;

$$\langle a\sqcup b\rangle X = \langle(\mathsf{FCD})(a\sqcup b)\rangle X = \langle(\mathsf{FCD})a\sqcup(\mathsf{FCD})b\rangle X = \langle(\mathsf{FCD})a\rangle X\sqcup\langle(\mathsf{FCD})b\rangle X = \langle a\rangle X\sqcup\langle b\rangle X;$$

thus it is a system of presides.

That this is a bounded system of presides follows from the formulas $(\mathsf{FCD})\bot^{\mathsf{RLD}(A,B)} = \bot$ and $(\mathsf{FCD})\top^{\mathsf{RLD}(A,B)} = \top$.

It is with identities, because proposition 991. It is with correct identities by proposition 951. □

FiXme: Also for pointfree reloids.

FiXme: These examples works for (dagger) systems of sides with binary product.

## 4. Product

Definition 1978. *Binary product* of objects of presides with identities is defined by the formula $X \times Y = \mathrm{id}_Y \circ \top \circ \mathrm{id}_X$.

Definition 1979. System of presides with identities is *with correct binary product* when $f \sqcap (X \times Y) = \mathrm{id}_Y \circ f \circ \mathrm{id}_X$ for every preside $f$.

Proposition 1980. $\langle A \times B\rangle X = \begin{cases} \bot & \text{if } X \asymp A \\ B & \text{if } X \not\asymp A \end{cases}$

Proof.

$$\langle A \times B\rangle X = \langle \mathrm{id}_B \circ \top \circ \mathrm{id}_A\rangle X = \langle \mathrm{id}_B\rangle\langle\top\rangle\langle\mathrm{id}_A\rangle X =$$
$$B \sqcap \langle\top\rangle(X \sqcap A) = B \sqcap \begin{cases} \bot & \text{if } X \asymp A \\ \top & \text{if } X \not\asymp A \end{cases} = \begin{cases} \bot & \text{if } X \asymp A \\ B & \text{if } X \not\asymp A \end{cases}$$

□

Definition 1981. I will call a system of sides *with correct meet* when

$$(X_0 \times Y_0) \sqcap (X_1 \times Y_1) = (X_0 \sqcap X_1) \times (Y_0 \sqcap Y_1).$$

Proposition 1982. Faithful systems of presides with identities are with correct meet.

Proof. $(X_0 \times Y_0) \sqcap (X_1 \times Y_1) = \mathrm{id}_{Y_1} \circ (X_0 \times Y_0) \circ \mathrm{id}_{X_1}$. Thus

$$\langle (X_0 \times Y_0) \sqcap (X_1 \times Y_1)\rangle P = \langle \mathrm{id}_{Y_1}\rangle\langle X_0 \times Y_0\rangle\langle\mathrm{id}_{X_1}\rangle P =$$
$$\langle\mathrm{id}_{Y_1}\rangle \begin{cases} \bot & \text{if } X_0 \asymp \langle\mathrm{id}_{X_1}\rangle P \\ Y_0 & \text{if } X_0 \not\asymp \langle\mathrm{id}_{X_1}\rangle P \end{cases} = \begin{cases} \bot & \text{if } X_0 \sqcap X_1 \asymp P \\ Y_0 \sqcap Y_1 & \text{if } X_0 \sqcap X_1 \not\asymp P \end{cases} =$$
$$\langle (X_0 \sqcap X_1) \times (Y_0 \sqcap Y_1)\rangle P.$$

So $(X_0 \times Y_0) \sqcap (X_1 \times Y_1) = (X_0 \sqcap X_1) \times (Y_0 \sqcap Y_1)$ follows by full faithfulness. □

Proposition 1983. Systems of presides with correct identities are with correct meet.

Proof. $(X_0 \times Y_0) \sqcap (X_1 \times Y_1) = \mathrm{id}_{Y_1} \circ (X_0 \times Y_0) \circ \mathrm{id}_{X_1} = \mathrm{id}_{Y_1} \circ (\mathrm{id}_{Y_0} \circ \top \circ \mathrm{id}_{X_0}) \circ \mathrm{id}_{X_1} = \mathrm{id}_{Y_0 \sqcap Y_1} \circ \top \circ \mathrm{id}_{X_0 \sqcap X_1} = (X_0 \sqcap X_1) \times (Y_0 \sqcap Y_1)$. □

For some sides holds the formula $f \circ (X \times Y) = X \times \langle f\rangle Y$. I refrain to give a name for this property.

## 5. Negative results

The following negative result generalizes theorem 3.8 in [3].

THEOREM 1984. The element $1^{(\mathrm{Src}\,\Upsilon)(\mathfrak{A},\mathfrak{A})}$ is not complemented if $\mathfrak{A}$ is a non-atomic boolean lattice, for every monotone system of sides.

PROOF. Let $T = 1^{(\mathrm{Src}\,\Upsilon)(\mathfrak{A},\mathfrak{A})}$.

Let's suppose $T \sqcup V = \top$ for $V \in (\mathrm{Src}\,\Upsilon)(\mathfrak{A},\mathfrak{A})$ and prove $T \sqcap V \neq \bot$.

Then $\langle T \sqcup V\rangle a = \top$ for all $a \neq \bot$ and thus $\langle V\rangle a \sqcup a = \top$.

Consequently $\langle V\rangle a \sqsupseteq \neg a$ for all $a \neq \bot$.

If $a$ isn't an atom, then there exists $b$ with $0 \sqsubset b \sqsubset a$ and hence $\langle V\rangle a \sqsupseteq \langle V\rangle b \sqsupseteq \neg b \sqsupset \neg a$; thus $\langle V\rangle a \sqsupset \neg a$.

There is such $c \sqsubset \top$ that $a \sqsubseteq c$ for every atom $a$. (Really, suppose some element $p \neq \bot$ has no atoms. Thus all atoms are in $\neg p$.)

For $a \not\sqsubseteq c$ we have $\langle V\rangle a \sqcap a \sqsupset \bot$ for all $a \sqsubseteq \neg c$ thus $\langle T \sqcap V\rangle a \sqsupseteq \langle V\rangle a \sqcap a \sqsupset \bot$. Thus $\langle (T \sqcap V) \circ \mathrm{id}_{\neg c}\rangle a \sqsupset \bot$

So $T \sqcap V \sqsupseteq (T \sqcap V) \circ \mathrm{id}_{\neg c} \sqsupset \bot$. So $V$ is not a complement of $T$. □

COROLLARY 1985. $(\mathrm{Src}\,\Upsilon)(\mathfrak{A},\mathfrak{A})$ is not boolean if $\mathfrak{A}$ is a non-atomic boolean lattice.

## 6. Dagger systems of sides

PROPOSITION 1986.

1°. For a partially ordered dagger category, each of Hom-set of which has least element, we have $\bot^\dagger = \bot$.

2°. For a partially ordered dagger category, each of Hom-set of which has greatest element, we have $\top^\dagger = \top$.

PROOF. $\forall f \in \mathrm{Hom}(A,B) : \bot^\dagger \sqsubseteq f \Leftrightarrow \forall f \in \mathrm{Hom}(A,B) : \bot \sqsubseteq f^\dagger \Leftrightarrow \forall f \in \mathrm{Hom}(A,B) : \bot \sqsubseteq f \Leftrightarrow 1$. Thus $\bot^\dagger$ is the least.

The other items is dual. □

DEFINITION 1987. *Dagger system of presides with identities* is system of presides with identities with category $\mathrm{Src}\,\Upsilon$ being a partially ordered dagger category and $(\mathrm{id}_X)^\dagger = \mathrm{id}_X$ for every $X$.

PROPOSITION 1988. For a system of sides we have $(X \times Y)^\dagger = Y \times X$.

PROOF. $(X \times Y)^\dagger = (\mathrm{id}_Y \circ \top \circ \mathrm{id}_X)^\dagger = \mathrm{id}_X^\dagger \circ \top^\dagger \circ \mathrm{id}_Y^\dagger = \mathrm{id}_X \circ \top \circ \mathrm{id}_Y = Y \times X$. □

FiXme: Which properties of pointfree funcoids can be generalized for sides?

CHAPTER 9

# Backward Funcoids

This is a preliminary partial draft.

Fix a family $\mathfrak{A}$ of posets.

DEFINITION 1989. Let $f$ be a staroid of filters $\mathfrak{F}(\mathfrak{A}_i)$ on boolean lattices $\mathfrak{A}_i$. *Backward funcoid* for the argument $k \in \operatorname{dom} \mathfrak{A}$ of $f$ is the funcoid $\operatorname{Back}(f, k)$ defined by the formula (for every $X \in \mathfrak{A}_k$)

$$\langle \operatorname{Back}(f, k) \rangle X = \left\{ \frac{L \in \prod_{i \in \operatorname{dom} \mathfrak{A}} \mathfrak{F}(\mathfrak{A}_i)}{X \in \langle f \rangle_k L} \right\}.$$

PROPOSITION 1990. Backward funcoid is properly defined.

PROOF. $\langle \operatorname{Back}(f, k) \rangle^* (X \sqcup Y) = \left\{ \frac{L \in \prod \mathfrak{A}}{X \sqcup Y \in \langle f \rangle_k L} \right\} = \left\{ \frac{L \in \prod \mathfrak{A}}{X \in \langle f \rangle_k L \vee Y \in \langle f \rangle_k L} \right\} = \left\{ \frac{L \in \prod \mathfrak{A}}{X \in \langle f \rangle_k L} \right\} \cup \left\{ \frac{L \in \prod \mathfrak{A}}{Y \in \langle f \rangle_k L} \right\} = \langle \operatorname{Back}(f, k) \rangle^* X \cup \langle \operatorname{Back}(f, k) \rangle^* Y$. □

OBVIOUS 1991. Backward funcoid is co-complete.

PROPOSITION 1992. If $f$ is a principal staroid then $\operatorname{Back}(f, k)$ is a complete funcoid.

PROOF. ?? □

PROPOSITION 1993. $f$ can be restored from $\operatorname{Back}(f, k)$ (for every fixed $k$).

PROOF. ?? □

PROPOSITION 1994. $f \mapsto \operatorname{Back}(f, k)$ is an order isomorphism $\mathsf{Strd}^{\mathfrak{A}} \to \mathsf{FCD}\left( \mathfrak{A}_k, \mathsf{Strd}^{(\operatorname{dom} \mathfrak{A}) \setminus \{k\}} \right)$.

PROOF. ?? □

CHAPTER 10

# Quasi-atoms

Definition 1995. *Quasi-atoms* funcoid $\mathscr{A}$ is the funcoid $A \to \operatorname{atoms}^{\mathfrak{A}} A$ defined by the formula $\langle \mathscr{A} \rangle^* X = \operatorname{atoms}^{\mathfrak{A}} X$.

This really defines a funcoid because $\operatorname{atoms}^{\mathfrak{A}} \bot = \emptyset$ and $\operatorname{atoms}^{\mathfrak{A}}(X \cup Y) = \operatorname{atoms}^{\mathfrak{A}} X \cup \operatorname{atoms}^{\mathfrak{A}} Y$.

Obvious 1996. $\mathscr{A}$ is a co-complete funcoid.

Proposition 1997. $\left\langle \mathscr{A}^{-1} \right\rangle^* Y = \bigsqcup Y$.

Proof. $Y \not\asymp \langle \mathscr{A} \rangle^* X \Leftrightarrow Y \not\asymp \operatorname{atoms}^{\mathfrak{A}} X \Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{A}} X, y \in Y : x \not\asymp y \Leftrightarrow \exists y \in Y : X \not\asymp y \Leftrightarrow$ (because $X$ is a principal filter) $\Leftrightarrow X \not\asymp \bigsqcup Y$. $\square$

Note $\langle \mathscr{A} \rangle^* \mathcal{X} = \bigsqcap^{\mathscr{F}}_{X \in \operatorname{up} \mathcal{X}} \operatorname{atoms}^{\mathfrak{A}} X$;

$\left\langle \mathscr{A}^{-1} \right\rangle^* \mathcal{Y} = \bigsqcap^{\mathscr{F}}_{Y \in \operatorname{up} \mathcal{Y}} \bigsqcup Y$ ($\mathcal{Y}$ is filter on the set of ultrafilters).

Can $\operatorname{atoms}^{\mathfrak{A}} \mathcal{X}$ be restored knowing $\langle \mathscr{A} \rangle \mathcal{X}$? Can $\bigsqcup \mathcal{Y}$ be restored knowing $\left\langle \mathscr{A}^{-1} \right\rangle \mathcal{X}$?

Proposition 1998. (Provided that $A$ is infinite) $\mathscr{A}$ is not complete.

Proof. Take a nonprincipal ultrafilter $x$. Then $\left\langle \mathscr{A}^{-1} \right\rangle^* \{x\} = \bigsqcup \{x\} = x$ is a nonprincipal filter. $\square$

Conjecture 1999. There is such filter $\mathcal{X}$ that $\langle \mathscr{A} \rangle^* \mathcal{X}$ is non-principal.

Does quasi-atoms funcoid define a more elegant replacement of $\operatorname{atoms}^{\mathfrak{A}}$? Does this concept have any use?

CHAPTER 11

# Cauchy Filters on Reloids

In this chapter I consider *low filters* on reloids, generalizing Cauchy filters on uniform spaces. Using low filters, I define Cauchy-complete reloids, generalizing complete uniform spaces.

FiXme: I forgot to note that Cauchy spaces induce topological (or convergence) spaces.

## 1. Preface

Replace `\langle ...\rangle` with `\supfun{...}` in LaTeX.

This is a preliminary partial draft.

To understand this article you need first look into my book [2].

http://math.stackexchange.com/questions/401989/what-are-interesting-properties-of-totally-bounded-uniform-spaces

http://ncatlab.org/nlab/show/proximity+space#uniform_spaces for a proof sketch that proximities correspond to totally bounded uniformities.

## 2. Low spaces

FiXme: Analyze http://link.springer.com/article/10.1007/s10474-011-0136-9 (“A note on Cauchy spaces”), http://link.springer.com/article/10.1007/BF00873992 (“Filter spaces”). It also contains references to some useful results, including (“On continuity structures and spaces of mappings” freely available at https://eudml.org/doc/16128) that the category FIL of filter spaces is isomorphic to the category of filter merotopic spaces (copy its definition).

DEFINITION 2000. A *lower set*[1] of filters on $U$ (a set) is a set $\mathscr{C}$ of filters on $U$, such that if $\mathcal{G} \sqsubseteq \mathcal{F}$ and $\mathcal{F} \in \mathscr{C}$ then $\mathcal{G} \in \mathscr{C}$.

REMARK 2001. Note that we are particularly interested in nonempty (= containing the improper filter) lower sets of filters. This does not match the traditional theory of Cauchy spaces (see below) which are traditionally defined as not containing empty set. Allowing them to contain empty set has some advantages:

- Meet of any lower filters is a lower filter.
- Some formulas become a little simpler.

DEFINITION 2002. I call *low space* a set together with a nonempty lower set of filters on this set. Elements of a (given) low space are called *Cauchy filters*.

DEFINITION 2003. $\mathrm{GR}(U, \mathscr{C}) = \mathscr{C}$; $\mathrm{Ob}(U, \mathscr{C}) = U$. $\mathrm{GR}(U, \mathscr{C})$ is read as *graph* of space $(U, \mathscr{C})$. I denote $\mathrm{Low}(U)$ the set of graphs of low spaces on the set $U$. Similarly I will denote its subsets $\mathbf{ASJ}(U)$, $\mathbf{CASJ}(U)$, $\mathbf{Cau}(U)$, $\mathbf{CCau}(U)$ (see below).

FiXme: Should use “space structure” instead of “graph of space”, to match customary terminology.

[1] Remember that our orders on filters is the reverse to set theoretic inclusion. It could be called an *upper* set in other sources.

Definition 2004. Introduce an order on graphs of low spaces and on low spaces: $\mathscr{C} \sqsubseteq \mathscr{D} \Leftrightarrow \mathscr{C} \subseteq \mathscr{D}$ and $(U, \mathscr{C}) \sqsubseteq (U, \mathscr{D}) \Leftrightarrow \mathscr{C} \sqsubseteq \mathscr{D}$.

Obvious 2005. Every set of low spaces on some set is partially ordered.

## 3. Almost sub-join-semilattices

Definition 2006. For a join-semilattice $\mathfrak{A}$, a *almost sub-join-semilattice* is such a set $S \in \mathscr{P}\mathfrak{A}$, that if $\mathcal{F}, \mathcal{G} \in S$ and $\mathcal{F} \not\asymp \mathcal{G}$ then $\mathcal{F} \sqcup \mathcal{G} \in S$.

Definition 2007. For a complete lattice $\mathfrak{A}$, a *completely almost sub-join-semilattice* is such a set $S \in \mathscr{P}\mathfrak{A}$, that if $\bigsqcap T \neq \bot^{\mathscr{F}(X)}$ then $\bigsqcup T \in S$ for every $T \in \mathscr{P}S$.

Obvious 2008. Every completely almost sub-join-semilattice is a almost sub-join-semilattice.

## 4. Cauchy spaces

Definition 2009. A *reflexive* low space is a low space $(U, \mathscr{C})$ such that $\forall x \in U :\uparrow^U \{x\} \in \mathscr{C}$.

Definition 2010. The *identity* low space $1^{\mathrm{Low}(U)}$ on a set $U$ is the low space with graph $\left\{ \frac{\uparrow^U \{x\}}{x \in U} \right\}$.

Obvious 2011. A low space $f$ is reflexive iff $f \sqsupseteq 1^{\mathrm{Low}(\mathrm{Ob}\, f)}$.

Definition 2012. *An almost sub-join space* is a low space whose graph is an almost sub-join-semilattice. I will denote such spaces as **ASJ**.

Definition 2013. *A completely almost sub-join space* is a low space whose graph is a completely almost sub-join-semilattice. I will denote such spaces as **CASJ**.

Definition 2014. A *precauchy space* (aka *filter space*) is a reflexive low space. I will denote such spaces as **preCau**.

Definition 2015. A *Cauchy space* is a precauchy space which is also an almost sub-join space. I will denote such spaces as **Cau**.

Definition 2016. A *completely Cauchy space* is a precauchy space which is also a completely almost sub-join space. I will denote such spaces as **CCau**.

Obvious 2017. Every completely Cauchy space is a Cauchy space.

Proposition 2018. $a \sqcup^{\left\{ \frac{\mathcal{X} \in \mathscr{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\}} b = a \sqcup b$ for $a, b \in \left\{ \frac{\mathcal{X} \in \mathscr{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\}$, provided that $\mathcal{F}$ is a proper fixed Cauchy filter on an almost sub-join space.

Proof. $\mathcal{F}$ is proper. So we have $a \sqcap b \sqsupseteq \mathcal{F} \neq \bot^{\mathscr{F}(X)}$. Thus $a \sqcup b$ is a Cauchy filter and so $a \sqcup b \in \left\{ \frac{\mathcal{X} \in \mathscr{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\}$. □

Proposition 2019. $\bigsqcup^{\left\{ \frac{\mathcal{X} \in \mathscr{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\}} S = \bigsqcup S$ for nonempty $S \in \mathscr{P}\left\{ \frac{\mathcal{X} \in \mathscr{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\}$, provided that $\mathcal{F}$ is a proper fixed Cauchy filter on a completely almost sub-join space.

Proof. $\mathcal{F}$ is proper. So for every nonempty $S \in \mathscr{P}\left\{ \frac{\mathcal{X} \in \mathscr{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\}$ we have $\bigsqcap S \sqsupseteq \mathcal{F} \neq \bot^{\mathscr{F}(X)}$. Thus $\bigsqcup S$ is a Cauchy filter and so $\bigsqcup S \in \left\{ \frac{\mathcal{X} \in \mathscr{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\}$. □

Corollary 2020. Every proper Cauchy filter is contained in a unique maximal Cauchy filter (for completely almost sub-join spaces).

PROOF. Let $\mathcal{F}$ be a proper Cauchy filter. Then $\bigsqcup^{\left\{\frac{\mathcal{X}\in\mathscr{C}}{\mathcal{X}\sqsupseteq\mathcal{F}}\right\}}\left\{\frac{\mathcal{X}\in\mathscr{C}}{\mathcal{X}\sqsupseteq\mathcal{F}}\right\}$ (existing by the above proposition) is the maximal Cauchy filter containing $\mathcal{F}$.

Suppose another maximal Cauchy filter $\mathcal{T}$ contains $\mathcal{F}$. Then $\mathcal{T}\in\left\{\frac{\mathcal{X}\in\mathscr{C}}{\mathcal{X}\sqsupseteq\mathcal{F}}\right\}$ and thus $\mathcal{T}=\bigsqcup^{\left\{\frac{\mathcal{X}\in\mathscr{C}}{\mathcal{X}\sqsupseteq\mathcal{F}}\right\}}\left\{\frac{\mathcal{X}\in\mathscr{C}}{\mathcal{X}\sqsupseteq\mathcal{F}}\right\}$. □

## 5. Relationships with symmetric reloids

FiXme: Also consider relationships with funcoids.

DEFINITION 2021. Denote $(\mathsf{RLD})_{\mathrm{Low}}(U,\mathscr{C})=\bigsqcup\left\{\frac{\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}}{\mathcal{X}\in\mathscr{C}}\right\}$.

DEFINITION 2022. $(\mathrm{Low})\nu$ (*low space* for endoreloid $\nu$) is a low space on $U$ such that

$$\mathrm{GR}(\mathrm{Low})\nu=\left\{\frac{\mathcal{X}\in\mathscr{F}(U)}{\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}\sqsubseteq\nu}\right\}.$$

THEOREM 2023. If $(U,\mathscr{C})$ is a low space, then $(U,\mathscr{C})=(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}}(U,\mathscr{C})$.

PROOF. If $\mathcal{X}\in\mathscr{C}$ then $\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}\sqsubseteq(\mathsf{RLD})_{\mathrm{Low}}(U,\mathscr{C})$ and thus $\mathcal{X}\in\mathrm{GR}(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}}(U,\mathscr{C})$. Thus $(U,\mathscr{C})\sqsubseteq(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}}(U,\mathscr{C})$.

Let's prove $(U,\mathscr{C})\sqsupseteq(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}}(U,\mathscr{C})$.

Let $\mathcal{A}\in\mathrm{GR}(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}}(U,\mathscr{C})$. We need to prove $\mathcal{A}\in\mathscr{C}$.

Really $\mathcal{A}\times^{\mathsf{RLD}}\mathcal{A}\sqsubseteq(\mathsf{RLD})_{\mathrm{Low}}(U,\mathscr{C})$. It is enough to prove that $\exists\mathcal{X}\in\mathscr{C}:\mathcal{A}\sqsubseteq\mathcal{X}$.

Suppose $\nexists\mathcal{X}\in\mathscr{C}:\mathcal{A}\sqsubseteq\mathcal{X}$.

For every $\mathcal{X}\in\mathscr{C}$ obtain $X_{\mathcal{X}}\in\mathcal{X}$ such that $X_{\mathcal{X}}\notin\mathcal{A}$ (if forall $X\in\mathcal{X}$ we have $X_{\mathcal{X}}\in\mathcal{A}$, then $\mathcal{X}\sqsupseteq\mathcal{A}$ what is contrary to our supposition).

It is now enough to prove $\mathcal{A}\times^{\mathsf{RLD}}\mathcal{A}\not\sqsubseteq\bigsqcup\left\{\frac{\uparrow^U X_{\mathcal{X}}\times^{\mathsf{RLD}}\uparrow^U X_{\mathcal{X}}}{\mathcal{X}\in\mathscr{C}}\right\}$.

Really, $\bigsqcup\left\{\frac{\uparrow^U X_{\mathcal{X}}\times^{\mathsf{RLD}}\uparrow^U X_{\mathcal{X}}}{\mathcal{X}\in\mathscr{C}}\right\}=\uparrow^{\mathsf{RLD}(U,U)}\bigcup\left\{\frac{\uparrow^U X_{\mathcal{X}}\times^{\mathsf{RLD}}\uparrow^U X_{\mathcal{X}}}{\mathcal{X}\in\mathscr{C}}\right\}$. So our claim takes the form $\bigcup\left\{\frac{\uparrow^U X_{\mathcal{X}}\times^{\mathsf{RLD}}\uparrow^U X_{\mathcal{X}}}{\mathcal{X}\in\mathscr{C}}\right\}\notin\mathrm{GR}(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{A})$ that is $\forall A\in\mathcal{A}:\bigcup\left\{\frac{\uparrow^U X_{\mathcal{X}}\times^{\mathsf{RLD}}\uparrow^U X_{\mathcal{X}}}{\mathcal{X}\in\mathscr{C}}\right\}\not\supseteq A\times A$ what is true because $X_{\mathcal{X}}\not\supseteq A$ for every $A\in\mathcal{A}$. □

REMARK 2024. The last theorem does not hold with $\mathcal{X}\times^{\mathsf{FCD}}\mathcal{X}$ instead of $\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}$ (take $\mathscr{C}=\left\{\frac{\{x\}}{x\in U}\right\}$ for an infinite set $U$ as a counter-example).

REMARK 2025. Not every symmetric reloid is in the form $(\mathsf{RLD})_{\mathrm{Low}}(U,\mathscr{C})$ for some Cauchy space $(U,\mathscr{C})$. The same Cauchy space can be induced by different uniform spaces. See http://math.stackexchange.com/questions/702182/different-uniform-spaces-having-the-same-set-of-cauchy-filters

PROPOSITION 2026.

1°. $(\mathrm{Low})f$ is reflexive iff endoreloid $f$ is reflexive.
2°. $(\mathsf{RLD})_{\mathrm{Low}}f$ is reflexive iff low space $f$ is reflexive.

PROOF.

1°. $f$ is reflexive $\Leftrightarrow 1^{\mathsf{RLD}}\sqsubseteq f\Leftrightarrow\forall x\in\mathrm{Ob}\,f:\uparrow(\{x\}\times\{x\})\sqsubseteq f\Leftrightarrow\forall x\in\mathrm{Ob}\,f:\uparrow\{x\}\times^{\mathsf{RLD}}\uparrow\{x\}\sqsubseteq f\Leftrightarrow\forall x\in\mathrm{Ob}\,f:\uparrow\{x\}\in(\mathrm{Low})f\Leftrightarrow(\mathrm{Low})f$ is reflexive.

2°. Let $f$ is reflexive. Then $\forall x\in\mathrm{Ob}\,f:\uparrow\{x\}\in f$; $\forall x\in\mathrm{Ob}\,f:\uparrow\{x\}\times^{\mathsf{RLD}}\uparrow\{x\}\sqsubseteq(\mathsf{RLD})_{\mathrm{Low}}f$; $\forall x\in\mathrm{Ob}\,f:\uparrow(\{x\}\times\{x\})\sqsubseteq(\mathsf{RLD})_{\mathrm{Low}}f$; $1^{\mathsf{RLD}}\sqsubseteq(\mathsf{RLD})_{\mathrm{Low}}f$.

Let now $(\mathsf{RLD})_{\mathrm{Low}}f$ be reflexive. Then $f=(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}}f$ is reflexive.

□

DEFINITION 2027. A *transitive* low space is such low space $f$ that $(\mathsf{RLD})_{\mathrm{Low}} f \circ (\mathsf{RLD})_{\mathrm{Low}} f = (\mathsf{RLD})_{\mathrm{Low}} f$.

REMARK 2028. The composition $(\mathsf{RLD})_{\mathrm{Low}} f \circ (\mathsf{RLD})_{\mathrm{Low}} f$ may be inequal to $(\mathsf{RLD})_{\mathrm{Low}} \mu$ for all low spaces $\mu$ (exercise!). Thus usefulness of the concept of transitive low spaces is questionable.

REMARK 2029. Every low space is “symmetric” in the sense that it corresponds to a symmetric reloid.

THEOREM 2030. $(\mathrm{Low})$ is the upper adjoint of $(\mathsf{RLD})_{\mathrm{Low}}$.

PROOF. We will prove $(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}} f \sqsupseteq f$ and $(\mathsf{RLD})_{\mathrm{Low}}(\mathrm{Low}) g \sqsubseteq g$ (that $(\mathrm{Low})$ and $(\mathsf{RLD})_{\mathrm{Low}}$ are monotone is obvious).

Really:

$$\mathrm{GR}(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}} f = \mathrm{GR}(\mathrm{Low}) \bigsqcup \left\{ \frac{\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X}}{\mathcal{X} \in \mathrm{GR} f} \right\} = \left\{ \frac{\mathcal{Y} \in \mathscr{F}\,\mathrm{Ob}(f)}{\mathcal{Y} \times^{\mathsf{RLD}} \mathcal{Y} \sqsubseteq \bigsqcup \left\{ \frac{\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X}}{\mathcal{X} \in \mathrm{GR} f} \right\}} \right\} \supseteq \left\{ \frac{\mathcal{Y} \in \mathscr{F}\,\mathrm{Ob}(f)}{\mathcal{Y} \in \mathrm{GR} f} \right\} = \mathrm{GR} f;$$

$$(\mathsf{RLD})_{\mathrm{Low}}(\mathrm{Low}) g = \bigsqcup \left\{ \frac{\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X}}{\mathcal{X} \in \mathrm{GR}(\mathrm{Low}) g} \right\} = \bigsqcup \left\{ \frac{\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X}}{\mathcal{X} \in \mathscr{F}(\mathrm{Ob}\, g), \mathcal{X} \times^{\mathsf{RLD}} \mathcal{X} \sqsubseteq g} \right\} \sqsubseteq g.$$ □

COROLLARY 2031.
1°. $(\mathsf{RLD})_{\mathrm{Low}} \bigsqcup S = \bigsqcup \langle (\mathsf{RLD})_{\mathrm{Low}} \rangle^* S$;
2°. $(\mathrm{Low}) \bigsqcap S = \bigsqcap \langle (\mathrm{Low}) \rangle^* S$.

Below it’s proved that $(\mathrm{Low})$ and $(\mathsf{RLD})_{\mathrm{Low}}$ can be restricted to completely almost sub-join spaces and symmetrically transitive reloids. Thus they preserve joins of (completely) almost sub-join spaces and meets of symmetrically transitive reloids. FiXme: Check. FiXme: Move it to be below the definition.

## 6. Lattices of low spaces

PROPOSITION 2032. $\mu \sqsubseteq \nu \Leftrightarrow \forall \mathcal{X} \in \mathrm{GR}\,\mu \exists \mathcal{Y} \in \mathrm{GR}\,\nu : \mathcal{X} \sqsubseteq \mathcal{Y}$ for low filter spaces (on the same set $U$).

PROOF.

$\Rightarrow$. $\mu \sqsubseteq \nu \Leftrightarrow \mathrm{GR}\,\mu \subseteq \mathrm{GR}\,\nu \Rightarrow \forall \mathcal{X} \in \mathrm{GR}\,\mu \exists \mathcal{Y} \in \mathrm{GR}\,\nu : \mathcal{X} = \mathcal{Y} \Rightarrow \forall \mathcal{X} \in \mathrm{GR}\,\mu \exists \mathcal{Y} \in \mathrm{GR}\,\nu : \mathcal{X} \sqsubseteq \mathcal{Y}$.

$\Leftarrow$. Let $\forall \mathcal{X} \in \mathrm{GR}\,\mu \exists \mathcal{Y} \in \mathrm{GR}\,\nu : \mathcal{X} \sqsubseteq \mathcal{Y}$. Take $\mathcal{X} \in \mathrm{GR}\,\mu$. Then $\exists \mathcal{Y} \in \mathrm{GR}\,\nu : \mathcal{X} \sqsubseteq \mathcal{Y}$. Thus $\mathcal{X} \in \mathrm{GR}\,\nu$. So $\mathrm{GR}\,\mu \subseteq \mathrm{GR}\,\nu$ that is $\mu \sqsubseteq \nu$.

□

OBVIOUS 2033.
1°. $(\mathsf{RLD})_{\mathrm{Low}}$ is an order embedding.
2°. $(\mathrm{Low})$ is an order homomorphism.

I will denote $\bigsqcup$, $\bigsqcap$, $\sqcup$, $\sqcap$ the lattice operations on low spaces or graphs of low spaces.

PROPOSITION 2034. $\bigsqcup S = \bigcup S$ for every set $S$ of graphs of low spaces on some set.

PROOF. It’s enough to prove that there is a low space $\mu$ such that $\mathrm{GR}\,\mu = \bigcup S$. In other words, it’s enough to prove that $\bigcup S$ is a nonempty lower set, but that’s obvious. FiXme: A little more detailed proof. □

PROPOSITION 2035. $\prod S = \left\{ \frac{\prod \operatorname{im} P}{P \in \prod_{X \in S} X} \right\}$ for every set $S$ of graphs of low spaces on some set.

PROOF. First prove that there is such low space $\mu$ that $\mu = \left\{ \frac{\prod \operatorname{im} P}{P \in \prod_{X \in S} X} \right\}$. In other words, we need to prove that $\left\{ \frac{\prod \operatorname{im} P}{P \in \prod_{X \in S} X} \right\}$ is a nonempty lower set. That it is nonempty is obvious. Let filter $\mathcal{G} \sqsubseteq \mathcal{F}$ and $\mathcal{F} \in \left\{ \frac{\prod \operatorname{im} P}{P \in \prod_{X \in S} X} \right\}$. Then $\mathcal{F} = \prod \operatorname{im} P$ for a $P \in \prod_{X \in S} X$ that is $P(X) \in X$ for every $X \in S$. Take $P' = (\mathcal{G} \sqcap) \circ P$. Then $P' \in \prod_{X \in S} X$ because $P'(X) \in X$ for every $X \in S$ and thus obviously $\mathcal{G} = \prod \operatorname{im} P'$ and thus $\mathcal{G} \in \left\{ \frac{\prod \operatorname{im} P}{P \in \prod_{X \in S} X} \right\}$. So such $\mu$ exists.

It remains to prove that $\mu$ is the greatest lower bound of $S$.

$\mu$ is a lower bound of $S$. Really, let $X \in S$ and $Y \in X$. Then exists $P \in \prod_{X \in S} X$ such that $P(X) = Y$ (taken into account that every $X$ is nonempty) and thus $\operatorname{im} P \ni Y$ and so $\prod \operatorname{im} P \sqsubseteq Y$, that is (proposition 2032) $\mu \sqsubseteq X$.

Let $\nu$ be a lower bound of $S$. It remains to prove that $\mu \sqsupseteq \nu$, that is $\forall Q \in \nu : Q = \prod \operatorname{im} P$ for some $P \in \prod_{X \in S} X$. Take $P = (\lambda X \in S : Q)$. This $P \in \prod_{X \in S} X$ because $Q \in X$ for every $X \in S$. □

COROLLARY 2036. $f \sqcap g = \left\{ \frac{F \sqcap G}{F \in f, G \in g} \right\}$ for every graphs $f$ and $g$ of low spaces (on some set).

### 6.1. Its subsets.

PROPOSITION 2037. The set of sub-join low spaces (on some fixed set) is meet-closed in the lattice of low spaces on a set.

PROOF. Let $f$, $g$ be graphs of almost sub-join spaces (on some fixed set), $f \sqcap g = \left\{ \frac{F \sqcap G}{F \in f, G \in g} \right\}$.

If $\mathcal{A}, \mathcal{B} \in f \sqcap g$ and $\mathcal{A} \not\asymp \mathcal{B}$, then $\mathcal{A}, \mathcal{B} \in f$ and $\mathcal{A}, \mathcal{B} \in g$. Thus $\mathcal{A} \sqcup \mathcal{B} \in f$ and $\mathcal{A} \sqcup \mathcal{B} \in g$ and so $\mathcal{A} \sqcup \mathcal{B} \in f \sqcap g$. □

COROLLARY 2038. The set of Cauchy spaces (on some fixed set), is meet-closed in the lattice of low spaces on a set.

PROPOSITION 2039. The set of completely almost sub-join spaces is meet-closed in the lattice of low spaces on a set.

PROOF. Let $S$ be a set of graphs of almost completely sub-join low spaces (on some fixed set). $\prod S = \left\{ \frac{\prod \operatorname{im} P}{P \in \prod_{X \in S} X} \right\}$.

If $\mathcal{A}, \mathcal{B} \in \prod S$ and $\mathcal{A} \not\asymp \mathcal{B}$, then $\mathcal{A}, \mathcal{B} \in X$ for every $X \in S$. Thus $\mathcal{A} \sqcup \mathcal{B} \in X$ and so $\mathcal{A} \sqcup \mathcal{B} \in \prod S$. □

COROLLARY 2040. The set of completely Cauchy spaces is meet-closed in the lattice of low spaces on a set.

From the above it follows:

OBVIOUS 2041. The following sets are complete lattices in our order:

1°. almost sub-join spaces, whose graphs are almost sub-join-semilattices;
2°. completely almost sub-join spaces;
3°. reflexive low spaces;
4°. precauchy spaces;

5°. Cauchy spaces;
6°. completely Cauchy spaces.

Denote $Z(f) = \left\{ \frac{F \sqcup G}{F \in f, G \in f, F \not\asymp G} \right\} \cup \{\bot\}$ for every set $f$ of filters (on some fixed set).

PROPOSITION 2042. $Z(f) \sqsupseteq f$ for every set $f$ of filters.

PROOF. Consider for $F \in f$ both cases $F = \bot$ and $F \neq \bot$. □

LEMMA 2043. For graphs of low spaces $f$, $g$ (on the same set)

$$Q = \bigcup S \cup Z\left(\bigcup S\right) \cup Z\left(Z\left(\bigcup S\right)\right) \cup \dots$$

is a graph of some almost sub-join space.

PROOF. That it is nonempty and a lower set of filters is obvious. It remains to prove that it is an almost sub-join-semilattice.

Let $\mathcal{A}, \mathcal{B} \in Q$ and $\mathcal{A} \not\asymp \mathcal{B}$. Then

$$\mathcal{A}, \mathcal{B} \in \underbrace{Z \dots Z}_{n \text{ times}}\left(\bigcup S\right)$$

for a natural $n$. Thus

$$\mathcal{A} \sqcup \mathcal{B} \in \underbrace{Z \dots Z}_{n+1 \text{ times}}\left(\bigcup S\right)$$

and so $\mathcal{A} \sqcup \mathcal{B} \in Q$. □

PROPOSITION 2044. Join on the lattice of graphs of almost sub-join spaces is described by the formula

$$\bigsqcup^{\mathbf{ASJ}} S = \bigcup S \cup Z\left(\bigcup S\right) \cup Z\left(Z\left(\bigcup S\right)\right) \cup \dots$$

PROOF. The right part of the above formula $\mu$ is a graph of an almost sub-join space (lemma).

That $\mu$ is an upper bound of $S$ is obvious.

It remains to prove that $\mu$ is the least upper bound.

Suppose $\nu$ is an upper bound of $S$. Then $\nu \supseteq \bigcup S$. Thus, because $\nu$ is an almost sub-join-semilattice, $Z(\nu) \subseteq \nu$, likewise $Z(Z(\nu)) \subseteq \nu$, etc. Consequently $Z(\bigcup S) \subseteq \nu$, $Z(Z(\bigcup S)) \subseteq \nu$, etc. So we have $\mu \sqsubseteq \nu$. □

PROPOSITION 2045. FiXme: Should be merged with the previous proposition.

$$\bigsqcup^{\mathbf{ASJ}} S = \left\{ \frac{F_0 \sqcup \dots \sqcup F_{n-1}}{F_0, \dots, F_{n-1} \in \bigcup S, F_0 \not\asymp F_1 \wedge F_1 \not\asymp F_2 \wedge \dots \wedge F_{n-2} \not\asymp F_{n-1} \text{ for } n \in \mathbb{N}} \right\}.$$

REMARK 2046. We take $F_0 \sqcup \dots \sqcup F_{n-1} = \bot$ for $n = 0$.

PROOF. Denote the right part of the above formula as $R$.

Suppose $F \in R$. Let's prove by induction that $F \in Q$. If $F = \bot$ that's obvious. Suppose we know that $F_0 \sqcup \dots \sqcup F_{n-1} \in Q$ that is for a natural $m$

$$F_0 \sqcup \dots \sqcup F_{n-1} \in \underbrace{Z \dots Z}_{m \text{ times}}\left(\bigcup S\right)$$

for $F_0, \dots, F_{n-1} \in \bigcup S$, $F_0 \not\asymp F_1 \wedge F_1 \not\asymp F_2 \wedge \dots \wedge F_{n-2} \not\asymp F_{n-1}$ and also $F_{n-1} \not\asymp F_n$. Then $F_0 \sqcup \dots \sqcup F_{n-1} \not\asymp F_n$ and thus $F_0 \sqcup \dots \sqcup F_{n-1} \sqcup F_n \in \underbrace{Z \dots Z}_{m+1 \text{ times}} (\bigcup S)$ that is $F_0 \sqcup \dots \sqcup F_{n-1} \sqcup F_n \in Q$. So $F \in Q$ for every $F \in R$.

Now suppose $F \in Q$ that is for a natural $m$
$$F \in \underbrace{Z \dots Z}_{m \text{ times}}\left(\bigcup S\right).$$
Let's prove by induction that $F = F_0 \sqcup \dots \sqcup F_{n-1}$ for some $F_0, \dots, F_{n-1} \in \bigcup S$ such that $F_0 \not\asymp F_1 \wedge F_1 \not\asymp F_2 \wedge \dots \wedge F_{n-2} \not\asymp F_{n-1}$. If $m = 0$ then $F \in \bigcup S$ and our promise is obvious. Let our statement holds for a natural $m$. Prove that it holds for
$$F' \in \underbrace{Z \dots Z}_{m+1 \text{ times}}\left(\bigcup S\right).$$
We have $F' = Z(F)$ for some $F = F_0 \sqcup \dots \sqcup F_{n-1}$ where $F_0 \not\asymp F_1 \wedge F_1 \not\asymp F_2 \wedge \dots \wedge F_{n-2} \not\asymp F_{n-1}$. The case $F' = \bot$ is easy. So we can assume $F' = A \sqcup B$ where $A, B \in F$ and $A \not\asymp B$. By the statement of induction $A = A_0 \sqcup \dots \sqcup A_{p-1}$, $B = B_0 \sqcup \dots \sqcup B_{q-1}$ for natural $p$ and $q$, where $A_0 \not\asymp A_1 \wedge A_1 \not\asymp A_2 \wedge \dots \wedge A_{p-2} \not\asymp A_{p-1}$, $B_0 \not\asymp B_1 \wedge B_1 \not\asymp B_2 \wedge \dots \wedge B_{n-2} \not\asymp B_{n-1}$. Take $j$ such that $A \not\asymp B_j$ and then take $i$ such that $A_i \not\asymp B_j$. Then (using symmetry of the relation $\not\asymp$) we have $(A_0 \not\asymp A_1 \wedge A_1 \not\asymp A_2 \wedge \dots \wedge A_{p-2} \not\asymp A_{p-1}) \wedge (A_{p-1} \not\asymp A_{p-2} \not\asymp \dots A_{i+1} \not\asymp A_i) \wedge A_i \not\asymp B_j \wedge (B_j \not\asymp B_{j-1} \wedge \dots \wedge B_1 \not\asymp B_0) \wedge (B_0 \not\asymp B_1 \wedge B_1 \not\asymp B_2 \wedge \dots \wedge B_{q-2} \not\asymp B_{q-1})$. So $F' = A \sqcup B$ is representable as the join of a finite sequence of filters with each adjacent pair of filters in this sequence being intersecting. That is $F' \in Q$. □

Proposition 2047. The lattice of Cauchy spaces (on some set) is a complete sublattice of the lattice of almost sub-join spaces.

Proof. It's obvious, taking into account obvious 2011. □

Denote $Z_\infty(f) = \left\{ \frac{\bigsqcup T}{T \in \mathscr{P} f \wedge \bigsqcap T \neq \bot} \right\} \cup \{\bot\}$.

Proposition 2048. $Z_\infty(f) \sqsupseteq f$.

Proof. Consider for $F \in f$ both cases $F = \bot$ and $F \neq \bot$. □

Lemma 2049. If $S$ is a set of graphs of low spaces, then
$$Q = \bigcup S \cup Z_\infty\left(\bigcup S\right) \cup Z_\infty\left(Z_\infty\left(\bigcup S\right)\right) \cup \dots$$
is a graph of a completely Cauchy space.

Proof. That it is nonempty and a lower set of filters is obvious. It remains to prove that it is a completely almost sub-join-semilattice.

Let $T \in \mathscr{P} Q$ and $\bigsqcap T \neq \bot$. Then
$$T \in \mathscr{P} \underbrace{Z_\infty \dots Z_\infty}_{n \text{ times}}\left(\bigcup S\right)$$
for a natural $n$. Thus
$$T \in \mathscr{P} \underbrace{Z_\infty \dots Z_\infty}_{n+1 \text{ times}}\left(\bigcup S\right)$$
and so $\bigsqcup T \in Q$. □

Proposition 2050. The lattice of completely Cauchy spaces (on some set) is a complete sublattice of the lattice of completely almost sub-join spaces.

Proof. It's obvious, taking into account obvious 2011. □

Proposition 2051. Join of a set $S$ on the lattice of graphs of completely almost sub-join-semilattice is described by the formula:
$$\bigsqcup^{\mathbf{CASJ}} S = \bigcup S \cup Z_\infty\left(\bigcup S\right) \cup Z_\infty\left(Z_\infty\left(\bigcup S\right)\right) \cup \dots$$

PROOF. The right part of the above formula $\mu$ is a graph of an almost sub-join space (lemma).

That $\mu$ is an upper bound of $S$ is obvious.

It remains to prove that $\mu$ is the least upper bound.

Suppose $\nu$ is an upper bound of $S$. Then $\nu \supseteq \bigcup S$. Thus, because $\nu$ is an almost sub-join-semilattice, $Z_\infty(\nu) \subseteq \nu$, likewise $Z_\infty(Z_\infty(\nu)) \subseteq \nu$, etc. Consequently $Z_\infty(\bigcup S) \subseteq \nu$, $Z_\infty(Z_\infty(\bigcup S)) \subseteq \nu$, etc. So we have $\mu \sqsubseteq \nu$. □

CONJECTURE 2052.

1°. $\bigsqcup^{\mathbf{CASJ}} S = \left\{ \begin{array}{l} \frac{\bigsqcup T_0 \sqcup \dots \sqcup \bigsqcup T_{n-1}}{n \in \mathbb{N}, T_0, \dots, T_{n-1} \in \bigcup S,} \\ \bigsqcap T_0 \neq \bot \wedge \dots \wedge \bigsqcap T_{n-1} \neq \bot, \\ \bigsqcup T_0 \not\asymp \bigsqcup T_1 \wedge \dots \wedge \bigsqcup T_{n-2} \not\asymp \bigsqcup T_{n-1}. \end{array} \right\}$;

2°. $\bigsqcup^{\mathbf{CASJ}} S = \left\{ \begin{array}{l} \frac{\bigsqcup T_0 \sqcup \bigsqcup T_1 \sqcup \dots}{T_0, T_1, \dots \in \bigcup S,} \\ \bigsqcap T_0 \neq \bot \wedge \bigsqcap T_2 \neq \bot \wedge \dots, \bigsqcup T_0 \not\asymp \bigsqcup T_1 \wedge \bigsqcup T_1 \not\asymp \bigsqcup T_2 \wedge \dots \end{array} \right\}$.

## 7. Up-complete low spaces

DEFINITION 2053. *Ideal base* is a nonempty subset $S$ of a poset such that $\forall a, b \in S \exists c \in S : (a, b \sqsubseteq c)$.

OBVIOUS 2054. Ideal base is dual of filter base.

THEOREM 2055. Product of nonempty posets is a ideal base iff every factor is an ideal base.

PROOF. FiXme: more detailed proof

In one direction it is easy: Suppose one multiplier is not a dcpo. Take a chain with fixed elements (thanks our posets are nonempty) from other multipliers and for this multiplier take the values which form a chain without the join. This proves that the product is not a dcpo.

Let now every factor is dcpo. $S$ is a filter base in $\prod \mathfrak{A}$ iff each component is a filter base. Each component has a join. Thus by proposition 618 $S$ has a componentwise join. □

DEFINITION 2056. I call a low space *up-complete* when each ideal base (or equivalently every nonempty chain, see theorem 567) in this space has join in this space.

REMARK 2057. Elements of this ideal base are filters. (Thus is could be called a generalized ideal base.)

EXAMPLE 2058.

1°. $\left\{ \frac{\mathcal{X} \in \mathfrak{F}]0; +\infty[}{\exists \varepsilon > 0 : \mathcal{X} \sqsubseteq \uparrow ]\varepsilon; +\infty[} \right\} \cup \uparrow \{0\}$ is a graph of Cauchy space on $\mathbb{R}_+$, but not up-complete.

2°. $\mathfrak{F}[0; +\infty[$ is a strictly greater graph of Cauchy space on $\mathbb{R}_+$ and is up-complete.

LEMMA 2059. Let $f$ be a reloid. Each ideal base $T \subseteq \left\{ \frac{(\mathcal{A},\mathcal{B})}{\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}\sqsubseteq f} \right\}$ has a join in this set.

PROOF. Let $T$ be an ideal base and $\forall(\mathcal{A},\mathcal{B}) \in T : \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \sqsubseteq f$.
$\forall(\mathcal{A},\mathcal{B}) \in T \forall \mathcal{X} \in \mathscr{F} \operatorname{Src} f : (\mathcal{X} \not\asymp \mathcal{A} \Rightarrow \mathcal{B} \sqsubseteq \langle f \rangle \mathcal{X})$;
taking join we have:
$\forall \mathcal{A} \in \Pr_0 T \forall \mathcal{X} \in \mathscr{F} \operatorname{Src} f : \left(\mathcal{X} \not\asymp \mathcal{A} \Rightarrow \bigsqcup_{\mathcal{B}\in\Pr_1 T} \mathcal{B} \sqsubseteq \langle f \rangle \mathcal{X}\right)$;
$\forall \mathcal{A} \in \Pr_0 T : \mathcal{A} \times^{\mathsf{FCD}} \bigsqcup_{\mathcal{B}\in\Pr_1 T} \mathcal{B} \sqsubseteq f$.
Now repeat a similar operation second time:
$\forall \mathcal{A} \in \Pr_0 T : \bigsqcup_{\mathcal{B}\in\Pr_1 T} \mathcal{B} \times^{\mathsf{FCD}} \mathcal{A} \sqsubseteq f^{-1}$;
$\forall \mathcal{A} \in \Pr_0 T \forall \mathcal{Y} \in \mathscr{F} \operatorname{Dst} f : \left(\mathcal{Y} \not\asymp \bigsqcup_{\mathcal{B}\in\Pr_1 T} \mathcal{B} \Rightarrow \mathcal{A} \sqsubseteq \langle f^{-1} \rangle \mathcal{Y}\right)$;
$\forall \mathcal{Y} \in \mathscr{F} \operatorname{Dst} f : \left(\mathcal{Y} \not\asymp \bigsqcup_{\mathcal{B}\in\Pr_1 T} \mathcal{B} \Rightarrow \bigsqcup_{\mathcal{A}\in\Pr_0 T} \mathcal{A} \sqsubseteq \langle f^{-1} \rangle \mathcal{Y}\right)$;
$\bigsqcup_{\mathcal{B}\in\Pr_1 T} \mathcal{B} \times^{\mathsf{FCD}} \bigsqcup_{\mathcal{A}\in\Pr_0 T} \mathcal{A} \sqsubseteq f^{-1}$;
$\bigsqcup_{\mathcal{A}\in\Pr_0 T} \mathcal{A} \times^{\mathsf{FCD}} \bigsqcup_{\mathcal{B}\in\Pr_1 T} \mathcal{B} \sqsubseteq f$. But $\bigsqcup_{\mathcal{A}\in\Pr_0 T} \mathcal{A} \times^{\mathsf{FCD}} \bigsqcup_{\mathcal{B}\in\Pr_1 T} \mathcal{B}$ is the join in consideration, because ideal base is ideal base in each argument. □

PROPOSITION 2060. A Cauchy space generated by an endoreloid is always up-complete.

PROOF. Let $f$ be an endoreloid. $\mathrm{GR}(\mathrm{Low})f = \left\{ \frac{\mathcal{X}\in\operatorname{Ob} f}{\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}\sqsubseteq f} \right\}$.
Let $T \subseteq \left\{ \frac{\mathcal{X}\in\operatorname{Ob} f}{\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}\sqsubseteq f} \right\}$ be an ideal base.

Then $N = \left\{ \frac{(\mathcal{F},\mathcal{F})}{\mathcal{F}\in T} \right\}$ is also an ideal base. Obviously $N \subseteq \left\{ \frac{(\mathcal{A},\mathcal{B})}{\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}\sqsubseteq f} \right\}$. Thus by the lemma it has a join in $\left\{ \frac{(\mathcal{A},\mathcal{B})}{\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}\sqsubseteq f} \right\}$. It's easy to see that this join is in $\left\{ \frac{(\mathcal{A},\mathcal{A})}{\mathcal{A}\in\operatorname{Ob} f,\mathcal{A}\times^{\mathsf{RLD}}\mathcal{A}\sqsubseteq f} \right\}$. Consequently $T$ has a join in $\left\{ \frac{\mathcal{X}\in\operatorname{Ob} f}{\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}\sqsubseteq f} \right\}$. □

It is long time known that (using our terminology) low space induced by a uniform space is a Cauchy space, but that it is complete and up-complete is probably first discovered by Victor Porton.

## 8. More on Cauchy filters

OBVIOUS 2061. Low filter on an endoreloid $\nu$ is a filter $\mathcal{F}$ such that

$$\forall U \in \mathrm{GR}\, f \exists A \in \mathcal{F} : A \times A \subseteq U.$$

REMARK 2062. The above formula is the standard definition of Cauchy filters on uniform spaces.

PROPOSITION 2063. If $\nu \sqsupseteq \nu \circ \nu^{-1}$ then every neighborhood filter is a Cauchy filter, that it

$$\nu \sqsupseteq \langle (\mathsf{FCD})\nu \rangle^* \{x\} \times^{\mathsf{RLD}} \langle (\mathsf{FCD})\nu \rangle^* \{x\}$$

for every point $x$.

PROOF. $\langle (\mathsf{FCD})\nu \rangle^* \{x\} \times^{\mathsf{RLD}} \langle (\mathsf{FCD})\nu \rangle^* \{x\} = \langle (\mathsf{FCD})\nu \rangle \uparrow^{\operatorname{Ob}\nu} \{x\} \times^{\mathsf{RLD}} \langle (\mathsf{FCD})\nu \rangle \uparrow^{\operatorname{Ob}\nu} \{x\} = \nu \circ (\uparrow^{\operatorname{Ob}\nu} \{x\} \times^{\mathsf{RLD}} \uparrow^{\operatorname{Ob}\nu} \{x\}) \circ \nu^{-1} = \nu \circ (\uparrow^{\mathsf{RLD}(\operatorname{Ob}\nu,\operatorname{Ob}\nu)} \{(x,x)\}) \circ \nu^{-1} \sqsubseteq \nu \circ \mathrm{id}^{\mathsf{RLD}(\operatorname{Ob}\nu,\operatorname{Ob}\nu)} \circ \nu^{-1} = \nu \circ \nu^{-1} \sqsubseteq \nu$. □

PROPOSITION 2064. If $\nu \sqsupseteq \nu \circ \nu^{-1}$ a filter converges (in $\nu$) to a point, it is a low filter, provided that every neighborhood filter is a low filter.

PROOF. Let $\mathcal{F} \sqsubseteq \langle (\mathsf{FCD})\nu \rangle^* \{x\}$. Then $\mathcal{F} \times^{\mathsf{RLD}} \mathcal{F} \sqsubseteq \langle (\mathsf{FCD})\nu \rangle^* \{x\} \times^{\mathsf{RLD}} \langle (\mathsf{FCD})\nu \rangle^* \{x\} \sqsubseteq \nu$. □

COROLLARY 2065. If a filter converges to a point, it is a low filter, provided that $\nu \sqsupseteq \nu \circ \nu^{-1}$.

## 9. Maximal Cauchy filters

LEMMA 2066. Let $S$ be a set of sets with $\bigsqcap\langle\uparrow^{\mathfrak{F}}\rangle^* S \neq 0^{\mathfrak{F}}$ (in other words, $S$ has finite intersection property). Let $T = \left\{\frac{X\times X}{X\in S}\right\}$. Then

$$\bigcup T \circ \bigcup T = \bigcup S \times \bigcup S.$$

PROOF. Let $x \in \bigcup S$. Then $x \in X$ for some $X \in S$. $\langle\bigcup T\rangle\{x\} \sqsupseteq \uparrow X \supseteq \bigcap S \neq \emptyset$. Thus $\langle\bigcup T \circ \bigcup T\rangle\{x\} = \langle\bigcup T\rangle\langle\bigcup T\rangle\{x\} \in \left\langle\uparrow^{\mathsf{FCD}}\bigcup T\right\rangle \bigsqcap\langle\uparrow^{\mathfrak{F}}\rangle S \sqsupseteq \bigsqcup\left\{\frac{\langle\uparrow^{\mathsf{FCD}}(X\times X)\rangle\bigsqcap\langle\uparrow^{\mathfrak{F}}\rangle S}{X\in S}\right\} = \bigsqcup\left\{\frac{\uparrow^{\mathfrak{F}} X}{X\in S}\right\} = \bigsqcup\langle\uparrow^{\mathfrak{F}}\rangle S$ that is $\langle\bigcup T\circ\bigcup T\rangle\{x\} \supseteq \bigcup S$. □

COROLLARY 2067. Let $S$ be a set of filters (on some fixed set) with nonempty meet. Let

$$T = \left\{\frac{\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}}{\mathcal{X}\in S}\right\}$$

Then

$$\bigsqcup T \circ \bigsqcup T = \bigsqcup S \times^{\mathsf{RLD}} \bigsqcup S.$$

PROOF. $\bigsqcup T \circ \bigsqcup T = \bigsqcap\left\{\frac{\uparrow^{\mathfrak{F}}(X\circ X)}{X\in\bigsqcup T}\right\}$.

If $X \in \bigsqcup T$ then $X = \bigcup_{Q\in T}(P_Q \times P_Q)$ where $P_Q \in Q$. Therefore by the lemma we have

$$\bigcup\left\{\frac{P_Q\times P_Q}{Q\in T}\right\}\circ\bigcup\left\{\frac{P_Q\times P_Q}{Q\in T}\right\} = \bigcup_{Q\in T} P_Q \times \bigcup_{Q\in T} P_Q.$$

Thus $X\circ X = \bigcup_{Q\in T} P_Q \times \bigcup_{Q\in T} P_Q$.

Consequently $\bigsqcup T\circ\bigsqcup T = \bigsqcap\left\{\frac{\uparrow^{\mathfrak{F}}\left(\bigcup_{Q\in T}P_Q\times\bigcup_{Q\in T}P_Q\right)}{X\in\bigsqcup T}\right\} \sqsupseteq \bigsqcup S\times^{\mathsf{RLD}}\bigsqcup S$.

$\bigsqcup T\circ\bigsqcup T \sqsubseteq \bigsqcup S\times^{\mathsf{RLD}}\bigsqcup S$ is obvious. □

DEFINITION 2068. I call an endoreloid $\nu$ *symmetrically transitive* iff for every symmetric endofuncoid $f \in \mathsf{FCD}(\operatorname{Ob}\nu, \operatorname{Ob}\nu)$ we have $f \sqsubseteq \nu \Rightarrow f\circ f \sqsubseteq \nu$.

OBVIOUS 2069. It is symmetrically transitive if at least one of the following holds:

1°. $\nu\circ\nu \sqsubseteq \nu$;
2°. $\nu\circ\nu^{-1} \sqsubseteq \nu$;
3°. $\nu^{-1}\circ\nu \sqsubseteq \nu$.
4°. $\nu^{-1}\circ\nu^{-1} \sqsubseteq \nu$.

COROLLARY 2070. Every uniform space is symmetrically transitive.

PROPOSITION 2071. (Low)$\nu$ is a completely Cauchy space for every symmetrically transitive endoreloid $\nu$.

PROOF. Suppose $S \in \mathscr{P}\left\{\frac{\mathcal{X}\in\mathfrak{F}}{\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}\sqsubseteq\nu}\right\}$.

$\bigsqcup\left\{\frac{\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}}{\mathcal{X}\in S}\right\} \sqsubseteq \nu$; $\bigsqcup\left\{\frac{\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}}{\mathcal{X}\in S}\right\}\circ\bigsqcup\left\{\frac{\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}}{\mathcal{X}\in S}\right\} \sqsubseteq \nu$; $\bigsqcup S\times^{\mathsf{RLD}}\bigsqcup S \sqsubseteq \nu$ (taken into account that $S$ has nonempty meet). Thus $\bigsqcup S$ is Cauchy. □

PROPOSITION 2072. The neighbourhood filter $\langle(\mathsf{FCD})\nu\rangle^*\{x\}$ of a point $x \in \operatorname{Ob}\nu$ is a maximal Cauchy filter, if it is a Cauchy filter and $\nu$ is a reflexive reloid. FiXme: Does it holds for all low filters?

PROOF. Let $\mathscr{N} = \langle(\mathsf{FCD})\nu\rangle^* \{x\}$. Let $\mathscr{C} \sqsupseteq \mathscr{N}$ be a Cauchy filter. We need to show $\mathscr{N} \sqsupseteq \mathscr{C}$.

Since $\mathscr{C}$ is Cauchy filter, $\mathscr{C} \times^{\mathsf{RLD}} \mathscr{C} \sqsubseteq \nu$. Since $\mathscr{C} \sqsupseteq \mathscr{N}$ we have $\mathscr{C}$ is a neighborhood of $x$ and thus $\uparrow^{\mathrm{Ob}\,\nu} \{x\} \sqsubseteq \mathscr{C}$ (reflexivity of $\nu$). Thus $\uparrow^{\mathrm{Ob}\,\nu} \{x\} \times^{\mathsf{RLD}} \mathscr{C} \sqsubseteq \mathscr{C} \times^{\mathsf{RLD}} \mathscr{C}$ and hence $\uparrow^{\mathrm{Ob}\,\nu} \{x\} \times^{\mathsf{RLD}} \mathscr{C} \sqsubseteq \nu$;

$$\mathscr{C} \sqsubseteq \operatorname{im}(\nu|_{\uparrow^{\mathrm{Ob}\,\nu}\{x\}}) = \langle(\mathsf{FCD})\nu\rangle^* \{x\} = \mathscr{N}.$$

□

## 10. Cauchy continuous functions

DEFINITION 2073. A function $f : U \to V$ is *Cauchy continuous* from a low space $(U, \mathscr{C})$ to a low space $(V, \mathscr{D})$ when $\forall \mathcal{X} \in \mathscr{C} : \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{X} \in \mathscr{D}$.

PROPOSITION 2074. Let $f$ be a principal reloid. Then $f \in \mathrm{C}((\mathsf{RLD})_{\mathrm{Low}}\mathscr{C}, (\mathsf{RLD})_{\mathrm{Low}}\mathscr{D})$ iff $f$ is Cauchy continuous.

$$\begin{aligned}
f \circ (\mathsf{RLD})_{\mathrm{Low}}\mathscr{C} \circ f^{-1} \sqsubseteq (\mathsf{RLD})_{\mathrm{Low}}\mathscr{D} \quad &\Leftrightarrow \\
\bigsqcup_{\mathcal{X}\in\mathscr{C}} (f \circ (\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X}) \circ f^{-1}) \sqsubseteq (\mathsf{RLD})_{\mathrm{Low}}\mathscr{D} \quad &\Leftrightarrow \\
\bigsqcup_{\mathcal{X}\in\mathscr{C}} (\langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{X} \times^{\mathsf{RLD}} \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{X}) \sqsubseteq (\mathsf{RLD})_{\mathrm{Low}}\mathscr{D} \quad &\Leftrightarrow \\
\forall \mathcal{X} \in \mathscr{C} : \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{X} \times^{\mathsf{RLD}} \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{X} \sqsubseteq (\mathsf{RLD})_{\mathrm{Low}}\mathscr{D} \quad &\Leftrightarrow \\
\forall \mathcal{X} \in \mathscr{C} : \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{X} \in \mathscr{D}.&
\end{aligned}$$

Thus we have expressed Cauchy properties through the algebra of reloids.

## 11. Cauchy-complete reloids

DEFINITION 2075. An endoreloid $\nu$ is *Cauchy-complete* iff every low filter for this reloid converges to a point.

REMARK 2076. In my book [2] *complete reloid* means something different. I will always prepend the word “Cauchy” to the word “complete” when meaning is by the last definition.

https://en.wikipedia.org/wiki/Complete_uniform_space#Completeness

## 12. Totally bounded

http://ncatlab.org/nlab/show/Cauchy+space

DEFINITION 2077. Low space is called *totally bounded* when every proper filter contains a proper Cauchy filter.

OBVIOUS 2078. A reloid $\nu$ is totally bounded iff

$$\forall X \in \mathscr{P}\,\mathrm{Ob}\,\nu \exists \mathcal{X} \in \mathfrak{F}^{\mathrm{Ob}\,\nu} : (\bot \neq \mathcal{X} \sqsubseteq \uparrow^{\mathrm{Ob}\,\nu} X \wedge \mathcal{X} \times^{\mathsf{RLD}} \mathcal{X} \sqsubseteq \nu).$$

THEOREM 2079. A symmetric transitive reloid is totally bounded iff its Cauchy space is totally bounded.

PROOF.

$\Rightarrow$. Let $\mathcal{F}$ be a proper filter on $\mathrm{Ob}\,\nu$ and let $a \in \operatorname{atoms}\mathcal{F}$. It's enough to prove that $a$ is Cauchy.

Let $D \in \mathrm{GR}\,\nu$. Let also $E \in \mathrm{GR}\,\nu$ is symmetric and $E \circ E \subseteq D$. There existsa finite subset $F \subseteq \mathrm{Ob}\,\nu$ such that $\langle E\rangle F = \mathrm{Ob}\,\nu$. Then obviously exists $x \in F$ such that $a \sqsubseteq \uparrow^{\mathrm{Ob}\,\nu} \langle E\rangle\{x\}$, but $\langle E\rangle\{x\} \times \langle E\rangle\{x\} = E^{-1} \circ (\{x\} \times \{x\}) \circ E \subseteq D$, thus $a \times^{\mathsf{RLD}} a \sqsubseteq \uparrow^{\mathsf{RLD}(\mathrm{Ob}\,\nu, \mathrm{Ob}\,\nu)} D$.

Because $D$ was taken arbitrary, we have $a \times^{\mathsf{RLD}} a \sqsubseteq \nu$ that is $a$ is Cauchy.

$\Leftarrow$. Suppose that Cauchy space associated with a reloid $\nu$ is totally bounded but the reloid $\nu$ isn't totally bounded. So there exists a $D \in \operatorname{GR} \nu$ such that $(\operatorname{Ob} \nu) \setminus \langle D \rangle F \neq \emptyset$ for every finite set $F$.

Consider the filter base

$$S = \left\{ \frac{(\operatorname{Ob} \nu) \setminus \langle D \rangle F}{F \in \mathscr{P} \operatorname{Ob} \nu, F \text{ is finite}} \right\}$$

and the filter $\mathcal{F} = \bigsqcap \langle \uparrow^{\operatorname{Ob} \nu} \rangle S$ generated by this base. The filter $\mathcal{F}$ is proper because intersection $P \cap Q \in S$ for every $P, Q \in S$ and $\emptyset \notin S$. Thus there exists a Cauchy (for our Cauchy space) filter $\mathcal{X} \sqsubseteq \mathcal{F}$ that is $\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X} \sqsubseteq \nu$.

Thus there exists $M \in \mathcal{X}$ such that $M \times M \subseteq D$. Let $F$ be a finite subset of $\operatorname{Ob} \nu$. Then $(\operatorname{Ob} \nu) \setminus \langle D \rangle F \in \mathcal{F} \sqsupseteq \mathcal{X}$. Thus $M \not\asymp (\operatorname{Ob} \nu) \setminus \langle D \rangle F$ and so there exists a point $x \in M \cap ((\operatorname{Ob} \nu) \setminus \langle D \rangle F)$.

$\langle M \times M \rangle \{p\} \subseteq \langle D \rangle \{x\}$ for every $p \in M$; thus $M \subseteq \langle D \rangle \{x\}$.

So $M \subseteq \langle D \rangle (F \cup \{x\})$. But this means that $M \in \mathcal{X}$ does not intersect $(\operatorname{Ob} \nu) \setminus \langle D \rangle (F \cup \{x\}) \in \mathcal{F} \sqsupseteq \mathcal{X}$, what is a contradiction (taken into account that $\mathcal{X}$ is proper).

□

http://math.stackexchange.com/questions/104696/pre-compactness-total-boundedness-and-cauchy-sequential-compactness

## 13. Totally bounded funcoids

Definition 2080. A funcoid $\nu$ is totally bounded iff

$$\forall X \in \operatorname{Ob} \nu \exists \mathcal{X} \in \mathfrak{F}^{\operatorname{Ob} \nu} : (0 \neq \mathcal{X} \sqsubseteq \uparrow^{\operatorname{Ob} \nu} X \wedge \mathcal{X} \times^{\mathsf{FCD}} \mathcal{X} \sqsubseteq \nu).$$

This can be rewritten in elementary terms (without using funcoidal product: $\mathcal{X} \times^{\mathsf{FCD}} \mathcal{X} \sqsubseteq \nu \Leftrightarrow \forall P \in \partial \mathcal{X} : \mathcal{X} \sqsubseteq \langle \nu \rangle P \Leftrightarrow \forall P \in \partial \mathcal{X}, Q \in \partial \mathcal{X} : P \ [\nu]^* \ Q \Leftrightarrow \forall P, Q \in \operatorname{Ob} \nu : (\forall E \in \mathcal{X} : (E \cap P \neq \emptyset \wedge E \cap Q \neq \emptyset) \Rightarrow P \ [\nu]^* \ Q)$.

Note that probably I am the first person which has written the above formula (for proximity spaces for instance) explicitly.

## 14. On principal low spaces

Definition 2081. A low space $(U, \mathscr{C})$ is *principal* when all filters in $\mathscr{C}$ are principal.

Proposition 2082. Having fixed a set $U$, principal reflexive low spaces on $U$ bijectively correspond to principal reflexive symmetric endoreloids on $U$.

Proof. ??
http://math.stackexchange.com/questions/701684/union-of-cartesian-squares □

## 15. Rest

https://en.wikipedia.org/wiki/Cauchy_filter#Cauchy_filters

https://en.wikipedia.org/wiki/Uniform_space "Hausdorff completion of a uniform space" here)

http://at.yorku.ca/z/a/a/b/13.htm : the category **Prox** of proximity spaces and proximally continuous maps (i.e. maps preserving nearness between two sets) is isomorphic to the category of totally bounded uniform spaces (and uniformly continuous maps).

https://en.wikipedia.org/wiki/Cauchy_space http://ncatlab.org/nlab/show/Cauchy+space

http://arxiv.org/abs/1309.1748

http://projecteuclid.org/download/pdf_1/euclid.pja/1195521991

http://www.emis.de/journals/HOA/IJMMS/Volume5_3/404620.pdf

~/math/books/Cauchy_spaces.pdf

https://ncatlab.org/nlab/show/Cauchy+space defines compact Cauchy spaces!

http://www.hindawi.com/journals/ijmms/1982/404620/abs/ (open access article) describes criteria for a Cauchy space to be uniformizable.

CHAPTER 12

# Funcoidal groups

REMARK 2083. FiXme: Move this into the book. If $\mu$ and $\nu$ are cocomplete endofuncoids, then we can describe $f \in \mathrm{C}(\mu, \nu)$ without using filters by the formulas:

1°. $\langle f\rangle^* \langle\mu\rangle^* X \sqsubseteq \langle\nu\rangle^* \langle f\rangle^* X$ (for every set $X$ in $\mathscr{P} \operatorname{Ob} \mu$)
2°. $\langle\mu\rangle^* X \sqsubseteq \langle f^{-1}\rangle^* \langle\nu\rangle^* \langle f\rangle^* X$ (for every set $X$ in $\mathscr{P} \operatorname{Ob} \mu$)
3°. $\langle f\rangle^* \langle\mu\rangle^* \langle f^{-1}\rangle^* Y \sqsubseteq \langle\nu\rangle^* Y$ (for every set $Y$ in $\mathscr{P} \operatorname{Ob} \nu$)

Funcoidal groups are modeled after topological groups (see Wikipedia) and are their generalization.

DEFINITION 2084. *Funcoidal group* is a group $G$ together with endofuncoid $\mu$ on $\operatorname{Ob} G$ such that

1°. $(y\cdot) \in \mathrm{C}(\mu; \mu)$ for every $y \in G$;
2°. $(\cdot x) \in \mathrm{C}(\mu; \mu)$ for every $x \in G$;
3°. $(x \mapsto x^{-1}) \in \mathrm{C}(\mu; \mu)$ for every $x \in G$.

PROPOSITION 2085. $t \mapsto y \cdot t \cdot x$ and $t \mapsto y \cdot t^{-1} \cdot x$ are continuous functions.

PROOF. As composition of continuous functions. □

OBVIOUS 2086. Composition of functions of the forms $t \mapsto y \cdot t \cdot x$ and $t \mapsto y \cdot t^{-1} \cdot x$ are also a function of one of these forms.

What is the purpose of the following (yet unproved) proposition? I don't know, but it looks curious.

PROPOSITION 2087. Let $E$ be a composition of functions of a form $\langle\mu\rangle^*$, $\langle y\cdot\rangle^*$, $\langle\cdot x\rangle^*$, $\langle{}^{-1}\rangle^*$ (where $x$ and $y$ vary arbitrarily) such that $\mu$ is met in the composition at least once. Let also either $\mu = \mu \circ \mu$ or $\mu$ is met exactly once in the product. There are such elements $x_0$, $y_0$ that either

1°. $(t \mapsto y_0 \cdot t \cdot x_0) \circ \langle\mu\rangle \sqsubseteq E \sqsubseteq \langle\mu\rangle \circ (t \mapsto y_0 \cdot t \cdot x_0)$;
2°. $(t \mapsto y_0 \cdot t^{-1} \cdot x_0) \circ \langle\mu\rangle \sqsubseteq E \sqsubseteq \langle\mu\rangle \circ (t \mapsto y_0 \cdot t^{-1} \cdot x_0)$.

PROOF. Using continuity a few times we prove that $E \sqsubseteq \langle\mu\rangle^* \circ \ldots \circ \langle\mu\rangle^* \circ f_n \circ \ldots \circ f_1$ where $f_i$ are functions of the forms $t \mapsto y \cdot t \cdot x$ or $t \mapsto y \cdot t^{-1} \cdot x$ for $n \in \mathbb{N}$. But $\langle\mu\rangle^* \circ \ldots \circ \langle\mu\rangle^* = \langle\mu\rangle^*$ by conditions and $f_n \circ \ldots \circ f_1$ is of the form $t \mapsto y \cdot t \cdot x$ or $t \mapsto y \cdot t^{-1} \cdot x$ by above proposition. $E \sqsubseteq \langle\mu\rangle \circ (t \mapsto y_0 \cdot t \cdot x_0)$ or $E \sqsubseteq \langle\mu\rangle \circ (t \mapsto y_0 \cdot t^{-1} \cdot x_0)$

The second inequalty is similar. Note that $x_0$ and $y_0$ are the same for the first and for the second item. □

$(G, \mu)$ vs $(G, \mu^{-1})$ are they isomorphic?
FiXme: We can also define reloidal groups.

## 1. On "Each regular paratopological group is completely regular" article

In this chapter I attempt to rewrite the paper [1] in more general setting of funcoids and reloids. I attempt to construct a "royal road" to finding proofs of statements of this paper and similar ones, what is important because we lose 60 years waiting for any proof.

**1.1. Definition of normality.** By definition (slightly generalizing the special case if $\mu$ is a quasi-uniform space from [1]) a pair of an endo-reloid $\mu$ and a complete funcoid $\nu$ (playing role of a generalization of a topological space) on a set $U$ is *normal* when

$$\left\langle \nu^{-1}\right\rangle^* A \sqsubseteq \left\langle \nu^{-1\circ}\right\rangle^* \left\langle \nu^{-1}\right\rangle^* \langle F\rangle^* A$$

for every entourage $F \in \operatorname{up}\mu$ of $\mu$ and every set $A \subseteq U$.

Note that this is *not* the same as customary definition of normal topological spaces.

Theorem 2088. An endoreloid $\mu$ is normal on endoreloid $\nu$ iff

$$\nu \circ \nu^{-1} \sqsubseteq \nu^{-1} \circ (\mathsf{FCD})\mu.$$

Proof. Equivalently transforming the criterion of normality (which should hold for all $F \in \operatorname{up}\mu$) using proposition 1932:

$\langle \nu\rangle^* \left\langle \nu^{-1}\right\rangle^* A \sqsubseteq \left\langle \nu^{-1}\right\rangle^* \langle F\rangle^* A$.

Also note

$\prod^{\mathscr{F}}_{F\in\operatorname{up}\mu} \left\langle \nu^{-1}\right\rangle^* \langle F\rangle^* A$ = (because funcoids preserve filtered meets) = $\left\langle \nu^{-1}\right\rangle^* \prod^{\mathscr{F}}_{F\in\operatorname{up}\mu} \langle F\rangle^* A = \left\langle \nu^{-1}\right\rangle^* \langle (\mathsf{FCD})\mu\rangle^* A$.

Thus the above is equivalent to $\langle \nu\rangle^* \left\langle \nu^{-1}\right\rangle^* A \sqsubseteq \left\langle \nu^{-1}\right\rangle^* \langle (\mathsf{FCD})\mu\rangle^* A$.

And this is in turn equivalent to

$$\nu \circ \nu^{-1} \sqsubseteq \nu^{-1} \circ (\mathsf{FCD})\mu.$$

□

Definition 2089. An endofuncoid $\mu$ is *normal* on endofuncoid $\nu$ when $\nu\circ\nu^{-1} \sqsubseteq \nu^{-1} \circ \mu$. FiXme: No need for $\nu$ to be endomorphism.

Obvious 2090.

1°. Endoreloid $\mu$ is normal on endofuncoid $\nu$ iff endofuncoid $(\mathsf{FCD})\mu$ is normal on endofuncoid $\nu$.

2°. Endofuncoid $\mu$ is normal on endoreloid $\nu$ iff endofuncoid $(\mathsf{RLD})_{\mathrm{in}}\mu$ is normal on endofuncoid $\nu$.

Corollary 2091. If $\nu$ is a symmetric endofuncoid and $\mu \sqsupseteq \nu^{-1}$, then it is normal.

Corollary 2092. (generalization of proposition 1 in [1]) If $\nu$ is a symmetric endofuncoid and $\operatorname{Compl}\mu \sqsupseteq \nu^{-1}$, then it is normal.

Definition 2093. A funcoid $\nu$ is *normally reloidazable* iff there exist a reloid $\mu$ such that $(\mu,\nu)$ is normal and $\nu = \operatorname{Compl}(\mathsf{FCD})\mu$.

Definition 2094. A funcoid $\nu$ is *normally quasi-uniformizable* iff there exist a quasi-uniform space (= reflexive and transitive reloid) $\mu$ such that $(\mu,\nu)$ is normal and $\nu = \operatorname{Compl}(\mathsf{FCD})\mu$.

Proposition 2095. A funcoid $\nu$ is normally reloidazable iff there exist a funcoid $\mu$ such that $\mu$ is normal on $\nu$ and $\nu = \operatorname{Compl}\mu$.

PROPOSITION 2096. A funcoid $\nu$ is normally quasi-uniformizable iff there exist a quasi-proximity space (= reflexive and transitive funcoid) $\mu$ such that $\mu$ is normal on $\nu$ and $\nu = \operatorname{Compl} \mu$.

PROOF. Obvious 2090 and the fact that (FCD) is an isomorphism between reflexive and transitive funcoids and reflexive and transitive reloids. □

In other words, it is normally reloidazable or normally quasi-uniformizable when

$$(\operatorname{Compl} \mu) \circ (\operatorname{Compl} \mu)^{-1} \sqsubseteq (\operatorname{Compl} \mu)^{-1} \circ \mu$$

for suitable $\mu$.

**1.2. Urysohn's lemma and friends.** For a detailed proof of Urysohn's lemma see also:
http://homepage.math.uiowa.edu/~jsimon/COURSES/M132Fall07/UrysohnLemma_v5.pdf
https://proofwiki.org/wiki/Urysohn's_Lemma
http://planetmath.org/proofofurysohnslemma

https://en.wikipedia.org/wiki/Proximity_space says that "The resulting topology is always completely regular. This can be proven by imitating the usual proofs of Urysohn's lemma, using the last property of proximal neighborhoods to create the infinite increasing chain used in proving the lemma."

Below follows an alternative proof of Urysohn lemma. *The proof was based on a conjecture proved false, see example 1271!*

LEMMA 2097. If $\langle\mu\rangle\mathcal{A} \asymp \mathcal{B}$ for a complete funcoid $\mu$ and $\mathcal{A}$, $\mathcal{B}$ are filters on relevant sets, then there exists $U \in \operatorname{up} \mu$ such that $\langle U\rangle\mathcal{A} \asymp \mathcal{B}$.

PROOF. Prove that $\left\{ \frac{\langle U\rangle\mathcal{A}}{U \in \operatorname{up} \mu} \right\}$ is a filter base. That it is nonempty is obvious.

Let $\mathcal{X}, \mathcal{Y} \in \left\{ \frac{\langle U\rangle\mathcal{A}}{U \in \operatorname{up} \mu} \right\}$. Then $\mathcal{X} = \langle U_{\mathcal{X}}\rangle\mathcal{A}$, $Y = \langle U_{\mathcal{Y}}\rangle\langle A\rangle$. Because $\mu$ is complete, we have (proposition 1049) $U_{\mathcal{X}} \sqcap U_{\mathcal{Y}} \in \operatorname{up} \mu$. Thus $\mathcal{X}, \mathcal{Y} \sqsupseteq \langle U_{\mathcal{X}} \sqcap U_{\mathcal{Y}}\rangle\mathcal{A} \in \left\{ \frac{\langle U\rangle\mathcal{A}}{U \in \operatorname{up} \mu} \right\}$.

Thus $\langle\mu\rangle\mathcal{A} \asymp \mathcal{B} \Leftrightarrow \mathcal{B} \sqcap \langle\mu\rangle\mathcal{A} = \bot \Leftrightarrow \exists U \in \operatorname{up} \mu : \mathcal{B} \sqcap \langle U\rangle\mathcal{A} = \bot \Leftrightarrow \exists U \in \operatorname{up} \mu : \langle U\rangle\mathcal{A} \asymp \mathcal{B}$. □

COROLLARY 2098. If $\langle\mu\rangle\mathcal{A} \asymp \langle\mu\rangle\mathcal{B}$ for a complete funcoid $\mu$ and $\mathcal{A}$, $\mathcal{B}$ are filters on relevant sets, then there exists $U \in \operatorname{up} \mu$ such that $\langle U\rangle\mathcal{A} \asymp \langle U\rangle\mathcal{B}$.

PROOF. Applying the lemma twice we can obtain $P, Q \in \operatorname{up} \mu$ such that $\langle P\rangle\mathcal{A} \asymp \langle Q\rangle\mathcal{B}$. But because $\mu$ is complete, we have $U = P \sqcap Q \in \operatorname{up} \mu$, while obviously $\langle U\rangle\mathcal{A} \asymp \langle U\rangle\mathcal{B}$. □

LEMMA 2099. (assuming conjecture 1271) For every $U \in \operatorname{up} \mu$ (where $\mu$ is a $T_4$ topological space) such that $\neg\left(A \left[U \circ U^{-1}\right]^* B\right)$ there is $W \in \operatorname{up} \mu$ such that $U \circ U^{-1} \sqsupseteq W \circ W^{-1} \circ W \circ W^{-1}$. For it holds $\neg\left(A \left[W \circ W^{-1}\right]^* B\right)$. We can assume that $\langle W\rangle^* X$ is open for every set $X$.

PROOF. $U \circ U^{-1} \in \operatorname{up}(\mu \circ \mu^{-1}) \subseteq \operatorname{up}(\mu \circ \mu^{-1} \circ \mu \circ \mu^{-1})$ (normality used). Thus by the conjecture there exists $W \in \operatorname{up} \mu$ such that $U \circ U^{-1} \sqsupseteq W \circ W^{-1} \circ W \circ W^{-1}$. $W \circ W^{-1} \sqsubseteq U \circ U^{-1}$ thus $\neg\left(A \left[W \circ W^{-1}\right]^* B\right)$.

To prove that $\langle W\rangle^* X$ is open for every set $X$, replace every $\langle W\rangle^*\{x\}$ with an open neighborhood $E \subseteq \langle W\rangle^* X$ of $\langle\mu\rangle^*\{x\}$ (and note that union of open sets is open). This new $W$ holds all necessary properties. □

LEMMA 2100. (assuming conjecture 1271) For every $U \in \operatorname{up}\mu$ (where $\mu$ is a $T_4$ topological space) such that $\neg\left(A \left[U \circ U^{-1}\right]^* B\right)$ there is $W \in \operatorname{up}\mu$ such that $U \circ U^{-1} \sqsupseteq \mu^{-1} \circ W \circ W^{-1} \circ W \circ W^{-1}$. For it holds $\neg\left(A \left[W \circ W^{-1}\right]^* B\right)$. We can assume that $\langle W\rangle^* X$ is open for every set $X$.

PROOF. Applying the previous lemma twice, we have some open $W \in \operatorname{up}\mu$ such that

$$U \circ U^{-1} \sqsupseteq W \circ W^{-1} \circ W \circ W^{-1} \circ W \circ W^{-1} \circ W \circ W^{-1}$$

and $\neg\left(A \left[W \circ W^{-1}\right]^* B\right)$. From this easily follows that

$$U \circ U^{-1} \sqsupseteq \mu^{-1} \circ W \circ W^{-1} \circ W \circ W^{-1}.$$

□

A modified proof of Urysohn’s lemma follows. This proof is in part based on [1]. (I attempt to find common generalization of Urysohn’s lemma and results from [1]).

$\mathbb{Q}_2 \stackrel{\text{def}}{=} \left\{\frac{k/2^n}{k,n\in\mathbb{N},0<k<2^n}\right\}$.

THEOREM 2101. Urysohn’s lemma (see Wikipedia) for disjoint closed sets $A$ and $B$ and function $f$ on a topological space $\mu$ (considered as complete funcoid).

PROOF. (assuming conjecture 1271) (used ProofWiki among other sources)

Because $A$ and $B$ are disjoint closed sets, we have $\langle\mu\rangle^* A \asymp \langle\mu\rangle^* B$. Thus by the corollary 2098 take $S_0 \in \operatorname{up}\mu$ and $\neg\left(A \left[S_0 \circ S_0^{-1}\right]^* B\right)$.

We have $\mu \circ \mu^{-1} \circ \mu \circ \mu^{-1} \sqsubseteq \mu \circ \mu^{-1}$ that is $\operatorname{up}(\mu \circ \mu^{-1} \circ \mu \circ \mu^{-1}) \supseteq \operatorname{up}(\mu \circ \mu^{-1})$.

Let’s prove by induction: There is a sequence $S$ of binary relations starting with $S_0$ such that $\neg\left(A \left[S_i \circ S_i^{-1}\right]^* B\right)$ and $S_i \circ S_i^{-1} \sqsupseteq \mu^{-1} \circ S_{i+1} \circ S_{i+1}^{-1} \circ S_{i+1} \circ S_{i+1}^{-1}$. It directly follows from the lemma (and uses the conjecture).

Denote $U_i = S_{i+1} \circ S_{i+1}^{-1}$. We have $U_i \sqsupseteq \mu^{-1} \circ U_{i+1} \circ U_{i+1}$ and $\neg\left(A \left[U_i\right]^* B\right)$.

By reflexivity of $\mu$ we have $U_{i+1} \subseteq U_{i+1} \circ U_{i+1} \subseteq U_i$.

Define fractional degree of $U$: $U^r \stackrel{\text{def}}{=} U_1^{r_1} \circ \ldots \circ U_{l_r}^{r_{l_r}}$ for every $r \in \mathbb{Q}_2$ where $r_1 \ldots r_{l_r}$ is the binary expansion of $r$.

Prove $U_r \subseteq U_0$. It is enough to prove $U_0 \supseteq U_1 \circ \ldots \circ U_{l_r}$. It follows from $U_2 \circ \ldots \circ U_{l_r} \subseteq U_1$, $U_3 \circ \ldots \circ U_{l_r} \subseteq U_2$, $\ldots$, $U_{l_r} \subseteq U_{l_r-1}$ what was shown above.

Let’s prove: For each $p, q \in \mathbb{Q}_2$ such that $p < q$ we have $\mu^{-1} \circ U^p \sqsubseteq U^q$. We can assume binary expansion of $p$ and $q$ be the same length $c$ (add zeros at the end of the shorter one). Now it is enough to prove

$$U_k \circ U_{k+1}^{q_{k+1}} \circ \cdots \circ U_c^{q_c} \sqsupseteq \mu^{-1} \circ U_{k+1}^{p_{k+1}} \circ U_{k+2}^{p_{k+2}} \circ \cdots \circ U_c^{p_c}.$$

But for this it’s enough

$$U_k \sqsupseteq \mu^{-1} \circ U_{k+1} \circ U_{k+2} \circ \cdots \circ U_c$$

what can be easily proved by induction: If $k = c$ then it takes the form $U_k \sqsupseteq \mu^{-1}$ what is obvious. Suppose it holds for $k$. Then $U_{k-1} \sqsupseteq \mu^{-1} \circ U_k \circ U_k \sqsupseteq \mu^{-1} \circ U_k \circ \mu^{-1} \circ U_{k+1} \circ U_{k+2} \circ \cdots \circ U_c \sqsupseteq \mu^{-1} \circ U_k \circ U_{k+1} \circ U_{k+2} \circ \cdots \circ U_c$, that is it holds for all natural $k \leq c$.

It is easy to prove that $\langle U^r\rangle^* X$ is open for every set $X$.

We have $\left\langle\mu^{-1}\right\rangle^* \langle U^p\rangle^* X \sqsubseteq \langle U^q\rangle^* X$.

$$f(z) \stackrel{\text{def}}{=} \inf\left(\{1\} \cup \left\{\frac{q \in \mathbb{Q}_2}{z \in \langle U^q\rangle^* A}\right\}\right).$$

$f$ is properly defined because $\{1\} \cup \left\{\frac{q\in\mathbb{Q}_2}{z\in\langle U^q\rangle^* A}\right\}$ is nonempty and bounded.

If $z \in A$ then $z \in \langle U^q \rangle^* A$ for every $q \in \mathbb{Q}_2$, thus $f(z) = 0$, because obviously $U^q \sqsupseteq 1$.

If $z \in B$ then $z \notin \langle U^q \rangle^* A$ for every $q \in \mathbb{Q}_2$, thus $f(z) = 1$, because $U^q \sqsubseteq U_0$.

It remains to prove that $f$ is continuous.

Let $D(x) = \{1\} \cup \left\{ \frac{q \in \mathbb{Q}_2}{z \in \langle U^q \rangle^* A} \right\}$.

To show that f is continuous, we first prove two smaller results:

(a) $x \in \langle \mu^{-1} \rangle^* \langle U^r \rangle^* A \Rightarrow f(x) \leq r$.

We have $x \in \langle \mu^{-1} \rangle^* \langle U^r \rangle^* A \Rightarrow \forall s > r : x \in \langle U^s \rangle^* A$, so $D(x)$ contains all rationals greater than $r$. Thus $f(x) \leq r$ by definition of $f$.

(b) $x \notin \langle U^r \rangle^* A \Rightarrow f(x) \geq r$.

We have $x \notin \langle U^r \rangle^* A \Rightarrow \forall s < r : x \notin \langle U^s \rangle^* A$. So $D(x)$ contains no rational less than $r$. Thus $f(x) \geq r$.

Let $x_0 \in S$ and let $]c; d[$ be an open real interval containing $f(x)$. We will find a neighborhood $T$ of $x_0$ such that $\langle f \rangle^* T \subseteq ]c; d[$.

Choose $p, q \in \mathbb{Q}$ such that $c < p < f(x_0) < q < d$. Let $T = \langle U^q \rangle^* A \setminus \langle \mu^{-1} \rangle^* \langle U^p \rangle^* A$.

Then since $f(x_0) < q$, we have that (b) implies vacuously that $x \in \langle U^q \rangle^* A$.

Since $f(x_0) > p$, (a) implies $x_0 \notin \langle U^p \rangle^* A$.

Hence $x_0 \in T$. Then $T$ is a neighborhood of $x_0$ because $T$ is open.

Finally, let $x \in T$.

Then $x \in \langle U^q \rangle^* A \subseteq \langle \mu^{-1} \rangle^* \langle U^q \rangle^* A$. So $f(x) \leq q$ by (a).

Also $x \notin \langle \mu^{-1} \rangle^* \langle U^p \rangle^* A$, so $x \notin \langle U^p \rangle^* A$ and $f(x) \geq p$ by (b).

Thus: $f(x) \in [p; q] \subseteq ]c; d[$.

Therefore $f$ is continuous.

Claim A: $f(x) > q \Rightarrow x \notin \langle \mu^{-1} \rangle^* \langle U^q \rangle^* A$

Claim B: $f(x) < q \Rightarrow x \in \langle U^q \rangle^* A$

Proof of claim A: If $f(x) > q$ then then there must be some gap between $q$ and $D(x)$; in particular, there exists some $q'$ such that $q < q' < f(x)$. But $q' < f(x) \Rightarrow x \notin \langle U^q \rangle^* A \Rightarrow x \notin \langle \mu^{-1} \rangle^* \langle U^q \rangle^* A$ (using that $\langle U^r \rangle^* X$ is open).

Proof of claim B: If $f(x) < q$ then there exists $q' \in D(x)$ such that $f(x) < q' < q$, in which case $q \in D(x)$, so $x \in \langle U^q \rangle^* A$.

To show that $f$ is continuous, it's enough to prove that preimages of $]a; 1]$ and $[0; a[$ are open.

Suppose $f(x) \in ]a; 1]$. Pick some $q$ with $a < q < f(x)$. We claim that the open set $W = X \setminus \langle f^{-1} \rangle^* \langle U^q \rangle^* A$ is a neighborhood of $x$ that is mapped by $f$ into $]a; 1]$. First, by (A), $f(x) > q \Rightarrow x \in W$, so $W$ is a neighborhood of $x$. If $y$ is any point of $W$, then $f(y)$ must be $\geq q > a$; otherwise, if $f(y) < q$, then, by (B) $y \in \langle U^q \rangle^* A \subseteq \langle f^{-1} \rangle^* \langle U^q \rangle^* A$.

Suppose $x \in f^{-1}[0; b[$ that is $f(x) < b$ and pick $q$ such that $f(x) < q < b$. By (B) $x \in \langle U^q \rangle^* A$. We claim that the neighborhood $\langle U^q \rangle^* A$ is mapped by $f$ into $[0; b[$. Suppose $y$ is any point of $\langle U^q \rangle^* A$. Then $q \in D(y)$, so $f(y) \leq q < b$. □

Theorem 2102. (from [1]) If $\mu$ is a normal quasi-uniformity on a topological space $\nu$, then for any nonempty subset $A \in \operatorname{Ob} \nu$ and entourage $U \in \operatorname{up} \mu$ there exists a continuous function $f : \operatorname{Ob} \nu \to [0; 1]$ such that $A \sqsubseteq \langle f^{-1} \rangle^* \{0\} \sqsubseteq \langle f^{-1} \rangle^* [0; 1[ \sqsubseteq \langle \nu^{-1\circ} \rangle^* \langle \nu^{-1} \rangle^* \langle U \rangle^* A$.

Proof. Choose inductively a sequence of entourages $(U_n)_{n=0}^{\infty}$ such that $U_0 = U$ and $U_{n+1} \circ U_{n+1} \sqsubseteq U_n$.

Denote $l_r = \max \left\{ \frac{n \in \mathbb{N}}{r_n = 1} \right\}$.

Define $U^r = U_{l_r}^{r_{l_r}} \circ \ldots \circ U_1^{r_1}$

Prove $\langle\nu^{-1}\rangle^*\langle U^q\rangle^* A \sqsubseteq \langle\nu^{-1}\circ\rangle^*\langle\nu^{-1}\rangle^*\langle U^r\rangle^* A$ for any $q < r$ in $\mathbb{Q}_2$. FiXme: Can be easily rewritten with the formula $\langle\nu\rangle^*\langle\nu^{-1}\rangle^*\langle U^q\rangle^* A \sqsubseteq \langle\nu^{-1}\rangle^*\langle U^r\rangle^* A$ instead. It may extend to non-complete funcoids.

There is such $l$ that $0 = q_l < r_l = 1$ and $q_i = r_i$ for all $i < l$.

It follows $l_q \neq l \leq l_r$.

Consider variants:

$l_q < l$. $\langle\nu^{-1}\rangle^*\langle U^q\rangle^* A \sqsubseteq \langle\nu^{-1}\rangle^*\left\langle U_{l_q}\circ\ldots\circ U_1^{q_1 q_{l_q}}\right\rangle^* A = \langle\nu^{-1}\rangle^*\left\langle U_{l_q}^{r_{l_q}}\circ\ldots\circ U_1^{r_1}\right\rangle^* A \sqsubseteq \langle\nu^{-1}\rangle^*\langle U_{l-1}^{r_{l-1}}\circ\ldots\circ U_1^{r_1}\rangle^* A \sqsubseteq \langle\nu^{-1}\circ\rangle^*\langle\nu^{-1}\rangle^*\langle U_l^{r_l}\circ U_{l-1}^{r_{l-1}}\circ\ldots\circ U_1^{r_1}\rangle^* A = \langle\nu^{-1}\circ\rangle^*\langle\nu^{-1}\rangle^*\langle U^r\rangle^* A$ (use $U_l^{r_l}\in\mathrm{up}(\mathsf{FCD})\mu$ by theorem 992).

$l < l_q$. Inclusions $U_k\circ U_k \sqsubseteq U_{k-1}$ for $l < k \leq l_q + 1$ guarantee that $U_{l_q+1}\circ U_{l_q}\circ\ldots\circ U_{l+1}\sqsubseteq U_l$ and then $\langle\nu^{-1}\rangle^*\langle U^q\rangle^* A \sqsubseteq \langle\nu^{-1}\rangle^*\left\langle U_{l_q}^{q_{l_q}}\circ\ldots\circ U_1^{q_1}\right\rangle^* A \sqsubseteq \langle\nu^{-1}\circ\rangle^*\langle\nu^{-1}\rangle^*\left\langle U_{l_q+1}^{q_{l_q+1}}\circ U_{l_q}^{q_{l_q}}\circ\ldots\circ U_1^{q_1}\right\rangle^* A = \langle\nu^{-1}\circ\rangle^*\langle\nu^{-1}\rangle^*\left\langle U_{l_q+1}\circ U_{l_q}^{q_{l_q}}\circ\ldots\circ U_l^0\circ\ldots\circ U_1^{q_1}\right\rangle^* A \sqsubseteq \langle\nu^{-1}\circ\rangle^*\langle\nu^{-1}\rangle^*\langle U_l\circ U_{l-1}^{q_{l-1}}\circ\ldots\circ U_1^{q_1}\rangle^* A \sqsubseteq \langle\nu^{-1}\circ\rangle^*\langle\nu^{-1}\rangle^*\langle U_l^{r_l}\circ U_{l-1}^{r_{l-1}}\circ\ldots\circ U_1^{r_1}\rangle^* A \sqsubseteq \langle\nu^{-1}\circ\rangle^*\langle\nu^{-1}\rangle^*\langle U_{l_r}^{r_{l_r}}\circ\ldots\circ U_1^{r_1}\rangle^* A = \langle\nu^{-1}\circ\rangle^*\langle\nu^{-1}\rangle^*\langle U_r\rangle^* A$.

Define $f$ by the formula $f(z) = \inf\left(\{1\}\cup\left\{\frac{q\in\mathbb{Q}_2}{z\in\langle\nu^{-1}\rangle^*\langle U^q\rangle^* A}\right\}\right)$.

It is clear?? that $A \sqsubseteq \langle f^{-1}\rangle^*\{0\}$ and $\langle f^{-1}\rangle^*[0;1[\sqsubseteq \bigcup_{q\in\mathbb{Q}_2}\langle\nu^{-1}\rangle^*\langle U^q\rangle^* A = \bigcup_{r\in\mathbb{Q}_2}\langle\nu^{-1}\circ\rangle^*\langle\nu^{-1}\rangle^*\langle U^r\rangle^* A \sqsubseteq \langle\nu^{-1}\circ\rangle^*\langle\nu^{-1}\rangle^*\langle U_0\rangle^* A$.

To prove that the map $f : X \to [0,1]$ is continuous, it suffices to check that for every real number $a\in]0;1[$ the sets $\langle f^{-1}\rangle^*[0;a[$ and $\langle f^{-1}\rangle^*]a;1]$ are open. This follows from the equalitites

$\langle f^{-1}\rangle^*[0;a[= \bigcup_{\mathbb{Q}_2\ni q<a}\langle\nu^{-1}\circ\rangle^*\langle\nu^{-1}\rangle^*\langle U^q\rangle^* A$ and $\langle f^{-1}\rangle^*]a;1] = \bigcup_{\mathbb{Q}_2\ni r>a}(X\setminus\langle\nu^{-1}\rangle^*\langle U^r\rangle^* A)$. □

How the formulas for normal ($T_4$) topological spaces and normal quasi-uniformities are related? Maybe this works: Replacing $\nu\to\mu\circ\mu^{-1}$, $\mu\to 1$ makes $\nu\circ\nu^{-1}\sqsubseteq\nu^{-1}\circ(\mathsf{FCD})\mu \to \mu\circ\mu^{-1}\circ\mu\circ\mu^{-1}\sqsubseteq\mu\circ\mu^{-1}$.

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