

Algebraic General Topology. Volume 1 addons

Victor Porton

E-mail address: porton@narod.ru

URL: <http://www.mathematics21.org>

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ABSTRACT. This file contains future addons for the free e-book “Algebraic
General Topology. Volume 1”, which are yet not enough ripe to be included
into the book.

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CHAPTER 1

About this document

This file contains future addons for the free e-book “Algebraic General Topology. Volume 1”, which are yet not enough ripe to be included into the book.

Theorem (including propositions, conjectures, etc.) numbers in this document start from the last theorem number in the book plus one. Theorems references inside this document are hyperlinked, but references to theorems in the book are not hyperlinked (because PDF viewer Okular 0.20.2 does not support Backward button after clicking a cross-document reference, and thus I want to avoid clicking such links).

Applications of algebraic general topology

1. “Hybrid” objects

Algebraic general topology allows to construct “hybrid” objects of “continuous” (as topological spaces) and discrete (as graphs).

Consider for example $D \sqcup T$ where D is a digraph and T is a topological space.

The n -th power $(D \sqcup T)^n$ yields an expression with 2^n terms. So treating $D \sqcup T$ as one object (what becomes possible using algebraic general topology) rather than the join of two objects may have an exponential benefit for simplicity of formulas.

2. A way to construct directed topological spaces

2.1. Some notation. I use \mathcal{E} and ι notations from `volume-2.pdf`. FiXme: Reorder document fragments to describe it before use.

I remind that $f|_X = f \circ \text{id}_X$ for binary relations, funcoids, and reloid.

$$f \parallel_X = f \circ (\mathcal{E}^X)^{-1}.$$

$$f \square X = \text{id}_X \circ f \circ \text{id}_X^{-1}.$$

As proved in `volume-2.pdf`, the following are bijections and moreover isomorphisms (for R being either funcoids or reloids or binary relations):

$$1^\circ. \left\{ \frac{(f|_X, f \parallel_X)}{f \in R} \right\};$$

$$2^\circ. \left\{ \frac{(f \square X, \iota_X f)}{f \in R} \right\}.$$

As easily follows from these isomorphisms and theorem 1091:

PROPOSITION 1849. For funcoids, reloids, and binary relations:

$$1^\circ. f \in C(\mu, \nu) \Rightarrow f \parallel_A \in C(\iota_A \mu, \nu);$$

$$2^\circ. f \in C'(\mu, \nu) \Rightarrow f \parallel_A \in C'(\iota_A \mu, \nu);$$

$$3^\circ. f \in C''(\mu, \nu) \Rightarrow f \parallel_A \in C''(\iota_A \mu, \nu).$$

2.2. Directed line and directed intervals. Let \mathfrak{A} be a poset. We will denote $\overline{\mathfrak{A}} = \mathfrak{A} \cup \{-\infty, +\infty\}$ the poset with two added elements $-\infty$ and $+\infty$, such that $+\infty$ is strictly greater than every element of \mathfrak{A} and $-\infty$ is strictly less.

FiXme: Generalize from \mathbb{R} to a wider class of posets.

DEFINITION 1850. For an element a of a poset \mathfrak{A}

$$1^\circ. J_{\geq}(a) = \left\{ \frac{x \in \mathfrak{A}}{x \geq a} \right\};$$

$$2^\circ. J_{>}(a) = \left\{ \frac{x \in \mathfrak{A}}{x > a} \right\};$$

$$3^\circ. J_{\leq}(a) = \left\{ \frac{x \in \mathfrak{A}}{x \leq a} \right\};$$

$$4^\circ. J_{<}(a) = \left\{ \frac{x \in \mathfrak{A}}{x < a} \right\}.$$

DEFINITION 1851. Let a be an element of a poset \mathfrak{A} .

$$1^\circ. \Delta(a) = \prod^{\mathcal{F}} \left\{ \frac{[x; y]}{x, y \in \mathfrak{A}, x < a \wedge y > a} \right\};$$

$$2^\circ. \Delta_{\geq}(a) = \prod^{\mathcal{F}} \left\{ \frac{[a; y]}{y \in \mathfrak{A}, y > a} \right\};$$

$$3^\circ. \Delta_{>}(a) = \prod^{\mathcal{F}} \left\{ \frac{[a; y]}{y \in \mathfrak{A}, x < a \wedge y > a} \right\};$$

$$4^\circ. \Delta_{\leq}(a) = \prod^{\mathcal{F}} \left\{ \frac{]x;a[}{x \in \mathfrak{A}, x < a} \right\};$$

$$5^\circ. \Delta_{<}(a) = \prod^{\mathcal{F}} \left\{ \frac{]x;a[}{x \in \mathfrak{A}, x < a} \right\}.$$

OBVIOUS 1852.

$$1^\circ. \Delta_{\geq}(a) = \Delta(a) \sqcap^{\mathcal{F}} @J_{\geq a};$$

$$2^\circ. \Delta_{>}(a) = \Delta(a) \sqcap^{\mathcal{F}} @J_{> a};$$

$$3^\circ. \Delta_{\leq}(a) = \Delta(a) \sqcap^{\mathcal{F}} @J_{\leq a};$$

$$4^\circ. \Delta_{<}(a) = \Delta(a) \sqcap^{\mathcal{F}} @J_{< a}.$$

DEFINITION 1853. Given a partial order \mathfrak{A} and $x \in \mathfrak{A}$, the following defines complete funcoids:

$$1^\circ. \langle |\mathfrak{A}| \rangle^* \{x\} = \Delta(x);$$

$$2^\circ. \langle |\mathfrak{A}|_{\geq} \rangle^* \{x\} = \Delta_{\geq}(x);$$

$$3^\circ. \langle |\mathfrak{A}|_{>} \rangle^* \{x\} = \Delta_{>}(x);$$

$$4^\circ. \langle |\mathfrak{A}|_{\leq} \rangle^* \{x\} = \Delta_{\leq}(x);$$

$$5^\circ. \langle |\mathfrak{A}|_{<} \rangle^* \{x\} = \Delta_{<}(x).$$

PROPOSITION 1854. The complete funcoid corresponding to the order topology¹ is equal to $|\mathfrak{A}|$.

PROOF. Because every open set is a finite union of open intervals, the complete funcoid f corresponding to the order topology is described by the formula: $\langle f \rangle^* \{x\} = \prod^{\mathcal{F}} \left\{ \frac{]a;b[}{a, b \in \mathfrak{A}, a < x \wedge b > x} \right\} = \Delta(x) = \langle |\mathfrak{A}| \rangle^* \{x\}$. Thus $f = |\mathfrak{A}|$. \square

EXERCISE 1855. Show that $|\mathfrak{A}|_{\geq}$ (in general) is not the same as “right order topology”².

PROPOSITION 1856.

$$1^\circ. \left\langle |\mathfrak{A}|_{\geq}^{-1} \right\rangle^* @X = @ \left\{ \frac{a \in \mathfrak{A}}{\forall y \in \mathfrak{A}: (y > a \Rightarrow X \cap]a;y[\neq \emptyset)} \right\};$$

$$2^\circ. \left\langle |\mathfrak{A}|_{>}^{-1} \right\rangle^* @X = @ \left\{ \frac{a \in \mathfrak{A}}{\forall y \in \mathfrak{A}: (y > a \Rightarrow X \cap]a;y[\neq \emptyset)} \right\};$$

$$3^\circ. \left\langle |\mathfrak{A}|_{\leq}^{-1} \right\rangle^* @X = @ \left\{ \frac{a \in \mathfrak{A}}{\forall x \in \mathfrak{A}: (x < a \Rightarrow X \cap]x;a[\neq \emptyset)} \right\};$$

$$4^\circ. \left\langle |\mathfrak{A}|_{<}^{-1} \right\rangle^* @X = @ \left\{ \frac{a \in \mathfrak{A}}{\forall x \in \mathfrak{A}: (x < a \Rightarrow X \cap]x;a[\neq \emptyset)} \right\}.$$

PROOF. $a \in \left\langle |\mathfrak{A}|_{\geq}^{-1} \right\rangle^* @X \Leftrightarrow @\{a\} \neq \left\langle |\mathfrak{A}|_{\geq}^{-1} \right\rangle^* @X \Leftrightarrow \langle |\mathfrak{A}|_{\geq} \rangle^* @\{a\} \neq @X \Leftrightarrow \Delta_{\geq}(a) \neq @X \Leftrightarrow \forall y \in \mathfrak{A}: (y > a \Rightarrow X \cap]a;y[\neq \emptyset)$.

$a \in \left\langle |\mathfrak{A}|_{>}^{-1} \right\rangle^* @X \Leftrightarrow @\{a\} \neq \left\langle |\mathfrak{A}|_{>}^{-1} \right\rangle^* @X \Leftrightarrow \langle |\mathfrak{A}|_{>} \rangle^* @\{a\} \neq @X \Leftrightarrow \Delta_{>}(a) \neq @X \Leftrightarrow \forall y \in \mathfrak{A}: (y > a \Rightarrow X \cap]a;y[\neq \emptyset)$.

The rest follows from duality. \square

REMARK 1857. On trivial ultrafilters these obviously agree:

$$1^\circ. \langle |\mathbb{R}|_{\geq} \rangle^* \{x\} = \langle |\mathbb{R}| \cap \geq \rangle^* \{x\};$$

$$2^\circ. \langle |\mathbb{R}|_{>} \rangle^* \{x\} = \langle |\mathbb{R}| \cap > \rangle^* \{x\};$$

$$3^\circ. \langle |\mathbb{R}|_{\leq} \rangle^* \{x\} = \langle |\mathbb{R}| \cap \leq \rangle^* \{x\};$$

$$4^\circ. \langle |\mathbb{R}|_{<} \rangle^* \{x\} = \langle |\mathbb{R}| \cap < \rangle^* \{x\}.$$

COROLLARY 1858.

$$1^\circ. |\mathbb{R}|_{\geq} = \text{Compl}(|\mathbb{R}| \cap \geq);$$

$$2^\circ. |\mathbb{R}|_{>} = \text{Compl}(|\mathbb{R}| \cap >);$$

$$3^\circ. |\mathbb{R}|_{\leq} = \text{Compl}(|\mathbb{R}| \cap \leq);$$

¹See Wikipedia for a definition of “Order topology”.

²See Wikipedia

4°. $|\mathbb{R}|_{<} = \text{Compl}(|\mathbb{R}| \cap <)$.

OBVIOUS 1859. **FiXme:** also what is the values of \setminus operation

1°. $|\mathbb{R}|_{\geq} = |\mathbb{R}|_{>} \sqcup 1;$

2°. $|\mathbb{R}|_{\leq} = |\mathbb{R}|_{<} \sqcup 1.$

3. Some inequalities

FiXme: Define the ultrafilter “at the left” and “at the right” of a real number. Also define “convergent ultrafilter”.

Denote $\Delta_{+\infty} = \prod_{x \in \mathbb{R}}]x; +\infty[$ and $\Delta_{-\infty} = \prod_{x \in \mathbb{R}}]-\infty; x[$.

The following proposition calculates $\langle \geq \rangle x$ and $\langle > \rangle x$ for all kinds of ultrafilters on \mathbb{R} :

PROPOSITION 1860.

1°. $\langle \geq \rangle \{\alpha\} =]\alpha; +\infty[$ and $\langle > \rangle \{\alpha\} =]\alpha; +\infty[$.

2°. $\langle \geq \rangle x = \langle > \rangle x =]\alpha; +\infty[$ for ultrafilter x at the right of a number α .

3°. $\langle \geq \rangle x = \langle > \rangle x = \Delta_{<}(\alpha) \sqcup]\alpha; +\infty[= \Delta_{\leq}(\alpha) \sqcup]\alpha; +\infty[$ for ultrafilter x at the left of a number α .

4°. $\langle \geq \rangle x = \langle > \rangle x = \Delta_{+\infty}$ for ultrafilter x at positive infinity.

5°. $\langle \geq \rangle x = \langle > \rangle x = \mathbb{R}$ for ultrafilter x at negative infinity.

PROOF.

1°. Obvious.

2°.

$$\begin{aligned} \langle \geq \rangle x &= \prod_{X \in \text{up } x}^{\mathcal{F}} \langle \geq \rangle (X \cap]\alpha; +\infty[) = \prod_{X \in \text{up } x}^{\mathcal{F}}]\alpha; +\infty[=]\alpha; +\infty[; \\ \langle > \rangle x &= \prod_{X \in \text{up } x}^{\mathcal{F}} \langle > \rangle (X \cap]\alpha; +\infty[) = \prod_{X \in \text{up } x}^{\mathcal{F}}]\alpha; +\infty[=]\alpha; +\infty[. \end{aligned}$$

3°. $\Delta_{<}(\alpha) \sqcup]\alpha; +\infty[= \Delta_{\leq}(\alpha) \sqcup]\alpha; +\infty[$ is obvious.

$$\langle > \rangle x = \prod_{X \in \text{up } x}^{\mathcal{F}} \langle > \rangle X \supseteq \prod_{X \in \text{up } x}^{\mathcal{F}} (\Delta_{<}(\alpha) \sqcup]\alpha; +\infty[) = \Delta_{<}(\alpha) \sqcup]\alpha; +\infty[$$

but $\langle \geq \rangle x \subseteq \Delta_{<}(\alpha) \sqcup]\alpha; +\infty[$ is obvious. It remains to take into account that $\langle > \rangle x \subseteq \langle \geq \rangle x$.

$$\begin{aligned} 4°. \quad \langle \geq \rangle x &= \prod_{X \in \text{up } x}^{\mathcal{F}} \langle \geq \rangle X = \prod_{X \in \text{up } x, \inf X \in X}^{\mathcal{F}} \langle \geq \rangle (X \cap]\alpha; +\infty[) = \\ &= \prod_{X \in \text{up } x}^{\mathcal{F}}]\inf X; +\infty[= \prod_{x > \alpha}^{\mathcal{F}}]x; +\infty[= \Delta_{+\infty}; \quad \langle > \rangle x = \prod_{X \in \text{up } x}^{\mathcal{F}} \langle > \rangle X = \\ &= \prod_{X \in \text{up } x, \inf X \in X}^{\mathcal{F}} \langle > \rangle (X \cap]\alpha; +\infty[) = \prod_{X \in \text{up } x}^{\mathcal{F}}]\inf X; +\infty[= \prod_{x > \alpha}^{\mathcal{F}}]x; +\infty[= \Delta_{+\infty}. \end{aligned}$$

5°. $\langle \geq \rangle x \supseteq \langle > \rangle x = \prod_{X \in \text{up } x}^{\mathcal{F}} \langle > \rangle X$ but $\langle > \rangle X =]-\infty; +\infty[$ for $X \in \text{up } x$ because X has arbitrarily small elements. □

LEMMA 1861. $\langle |\mathbb{R}| \rangle x \subseteq \langle > \rangle x = \langle \geq \rangle x$ for every nontrivial ultrafilter x .

PROOF. $\langle > \rangle x = \langle \geq \rangle x$ follows from the previous proposition.

$$\langle |\mathbb{R}| \rangle x = \prod_{X \in \text{up } x} \langle |\mathbb{R}| \rangle X = \prod_{X \in \text{up } x} \bigsqcup_{y \in X} \Delta(y).$$

Consider cases:

x is an ultrafilter at the right of some number α .

$$\begin{aligned} \langle |\mathbb{R}| \rangle x &= \prod_{X \in \text{up } x} \bigsqcup_{y \in X \cap]\alpha; +\infty[} \Delta(y) \subseteq]\alpha; +\infty[= \langle \geq \rangle x \text{ because} \\ &\bigsqcup_{y \in X \cap]\alpha; +\infty[} \Delta(y) \subseteq]\alpha; +\infty[. \end{aligned}$$

x is an ultrafilter at the left of some number α .

$\langle |\mathbb{R}| \rangle x \sqsubseteq \Delta(\alpha)$ is obvious. But $\langle \geq \rangle x \sqsupseteq \Delta(\alpha)$.

x is an ultrafilter at positive infinity.

$\langle |\mathbb{R}| \rangle x \sqsubseteq \Delta_{+\infty}$ is obvious. But $\langle \geq \rangle x = \Delta_{+\infty}$.

x is an ultrafilter at negative infinity.

Because $\langle \geq \rangle x = \mathbb{R}$.

□

COROLLARY 1862. $\langle |\mathbb{R}| \sqcap \geq \rangle x = \langle |\mathbb{R}| \rangle x$ for every nontrivial ultrafilter x .

PROOF. $\langle |\mathbb{R}| \sqcap \geq \rangle x = \langle |\mathbb{R}| \rangle \sqcap \langle \geq \rangle x = \langle |\mathbb{R}| \rangle x$.

□

So $\langle |\mathbb{R}| \sqcap \geq \rangle$ and $\langle |\mathbb{R}| \rangle$ agree on all ultrafilters except trivial ones.

PROPOSITION 1863. $|\mathbb{R}|_{>} \sqcap > = |\mathbb{R}|_{>} \sqsupseteq |\mathbb{R}|_{>}$.

PROOF. $|\mathbb{R}|_{>} \sqsubseteq >$ because $\langle |\mathbb{R}|_{>} \rangle^* x \sqsubseteq \langle > \rangle^* x$ and $|\mathbb{R}|_{>}$ is a complete funcoid.

□

LEMMA 1864. $\langle |\mathbb{R}|_{>} \rangle x \sqsubset \langle |\mathbb{R}|_{\geq} \rangle x$ for a nontrivial ultrafilter x .

PROOF. It enough to prove $\langle |\mathbb{R}|_{>} \rangle x \neq \langle |\mathbb{R}|_{\geq} \rangle x$.

Take x be an ultrafilter with limit point 0 on $\text{im } z$ where z is the sequence $n \mapsto \frac{1}{n}$.

$$\langle |\mathbb{R}|_{>} \rangle x \sqsubseteq \langle |\mathbb{R}|_{>} \rangle^* \text{im } z = \bigsqcup_{n \in \text{im } z} \Delta_{>} \left(\frac{1}{n} \right) \sqsubseteq \bigsqcup_{n \in \text{im } z} \left[\frac{1}{n}; \frac{1}{n-1} - \frac{1}{n} \right] \asymp \text{im } z.$$

Thus $\langle |\mathbb{R}|_{>} \rangle x \asymp \text{im } z$. But $\langle |\mathbb{R}|_{\geq} \rangle x \sqsubseteq \langle = \rangle x \not\asymp \text{im } z$.

□

COROLLARY 1865. $|\mathbb{R}|_{>} \sqsubset |\mathbb{R}|_{\geq}$.

PROPOSITION 1866. $|\mathbb{R}|_{>} \sqsubset |\mathbb{R}|_{\geq} \sqcap >$.

PROOF. It's enough to prove $|\mathbb{R}|_{>} \neq |\mathbb{R}|_{\geq} \sqcap >$.

Really, $\langle |\mathbb{R}|_{\geq} \sqcap > \rangle x = \langle |\mathbb{R}|_{\geq} \rangle x \neq \langle |\mathbb{R}|_{>} \rangle x$ (lemma).

□

PROPOSITION 1867.

- 1°. $|\mathbb{R}|_{\geq} \circ |\mathbb{R}|_{\geq} = |\mathbb{R}|_{\geq}$;
- 2°. $|\mathbb{R}|_{>} \circ |\mathbb{R}|_{>} = |\mathbb{R}|_{>}$;
- 3°. $|\mathbb{R}|_{\geq} \circ |\mathbb{R}|_{>} = |\mathbb{R}|_{>}$;
- 4°. $|\mathbb{R}|_{>} \circ |\mathbb{R}|_{\geq} = |\mathbb{R}|_{>}$.

PROOF. ??

□

CONJECTURE 1868.

- 1°. $(|\mathbb{R}| \sqcap \geq) \circ (|\mathbb{R}| \sqcap \geq) = |\mathbb{R}| \sqcap \geq$.
- 2°. $(|\mathbb{R}| \sqcap >) \circ (|\mathbb{R}| \sqcap >) = |\mathbb{R}| \sqcap >$.

4. Continuity

I will say that a property holds on a filter \mathcal{A} iff there is $A \in \text{up } \mathcal{A}$ on which the property holds.

Fixme: $f \in C(A, B) \wedge f \in C(\iota_A |\mathbb{R}|_{\geq}, \iota_B |\mathbb{R}|_{\geq}) \Leftrightarrow (f, f) \in C((A, \iota_A |\mathbb{R}|_{\geq}), (B, \iota_B |\mathbb{R}|_{\geq}))$

LEMMA 1869. Let function $f : A \rightarrow B$ where $A, B \in \mathcal{P}\mathbb{R}$ and A is connected.

- 1°. f is monotone and $f \in C(A, B)$ iff $f \in C(A, B) \cap C(\iota_A |\mathbb{R}|_{\geq}, \iota_B |\mathbb{R}|_{\geq})$
iff $f \in C(A, B) \cap C(\iota_A |\mathbb{R}|_{>}, \iota_B |\mathbb{R}|_{>})$ iff $f \in C(\iota_A |\mathbb{R}|_{\geq}, \iota_B |\mathbb{R}|_{\geq}) \cap C(\iota_A |\mathbb{R}|_{\leq}, \iota_B |\mathbb{R}|_{\leq})$.

2°. f is strictly monotone and $\in C(A, B)$ iff $f \in C(A, B) \cap C(\iota_A|\mathbb{R}|_>, \iota_B|\mathbb{R}|_>)$
 iff $f \in C(\iota_A|\mathbb{R}|_>, \iota_B|\mathbb{R}|_>) \cap C(\iota_A|\mathbb{R}|_<, \iota_B|\mathbb{R}|_<)$.

FiXme: Generalize for arbitrary posets. **FiXme:** Generalize for f being a funcoïd.

PROOF. Because f is continuous, we have $\langle f \circ \iota_A|\mathbb{R}| \rangle^* \{x\} \sqsubseteq \langle \iota_B|\mathbb{R}| \circ f \rangle^* \{x\}$
 that is $\langle f \rangle^* \Delta(x) \sqsubseteq \Delta(f(x))$ for every x .

If f is monotone, we have $\langle f \rangle^* \Delta_{\geq}(x) \sqsubseteq [f(x); \infty[$. Thus
 $\langle f \rangle^* \Delta_{\geq}(x) \sqsubseteq \Delta_{\geq}(f(x))$, that is $\langle f \circ \iota_A|\mathbb{R}|_{\geq} \rangle^* \{x\} \sqsubseteq \langle \iota_B|\mathbb{R}|_{\geq} \circ f \rangle^* \{x\}$, thus
 $f \in C(\iota_A|\mathbb{R}|_{\geq}, \iota_B|\mathbb{R}|_{\geq})$.

If f is strictly monotone, we have $\langle f \rangle^* \Delta_{>}(x) \sqsubseteq]f(x); \infty[$. Thus
 $\langle f \rangle^* \Delta_{>}(x) \sqsubseteq \Delta_{>}(f(x))$, that is $\langle f \circ \iota_A|\mathbb{R}|_{>} \rangle^* \{x\} \sqsubseteq \langle \iota_B|\mathbb{R}|_{>} \circ f \rangle^* \{x\}$, thus $f \in$
 $C(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{>})$.

Let now $f \in C(\iota_A|\mathbb{R}|_{\geq}, \iota_B|\mathbb{R}|_{\geq})$.

Take any $a \in A$ and let $c = \left\{ \frac{b \in B}{b \geq a, \forall x \in [a; b[: f(x) \geq f(a)} \right\}$. It's enough to prove that
 c is the right endpoint (finite or infinite) of A .

Indeed by continuity $f(a) \leq f(c)$ and if c is not already the right endpoint
 of A , then there is $b' > c$ such that $\forall x \in [c; b'[: f(x) \geq f(c)$. So we have $\forall x \in [a; b'[:$
 $f(x) \geq f(c)$ what contradicts to the above.

So f is monotone on the entire A .

$f \in C(\iota_A|\mathbb{R}|_{\geq}, \iota_B|\mathbb{R}|_{\geq}) \Rightarrow f \in C(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{\geq})$ is obvious. Reversely $f \in$
 $C(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{\geq}) \Rightarrow f \circ \iota_A|\mathbb{R}|_{>} \sqsubseteq \iota_B|\mathbb{R}|_{\geq} \circ f \Leftrightarrow \forall x \in \mathbb{R} : \langle f \rangle \langle \iota_A|\mathbb{R}|_{>} \rangle^* \{x\} \sqsubseteq$
 $\langle \iota_B|\mathbb{R}|_{\geq} \rangle^* \langle f \rangle^* \{x\} \Leftrightarrow \forall x \in \mathbb{R} : \langle f \rangle \Delta_{>}(x) \sqsubseteq \Delta_{\geq} f(x) \Leftrightarrow \forall x \in \mathbb{R} : \langle f \rangle \Delta_{>}(x) \sqcup$
 $\{f(x)\} \sqsubseteq \Delta_{\geq} f(x) \Leftrightarrow \forall x \in \mathbb{R} : \langle f \rangle \Delta_{>}(x) \sqcup \{x\} \sqsubseteq \Delta_{\geq} f(x) \Leftrightarrow \forall x \in \mathbb{R} : \langle f \rangle \Delta_{>}(x) \sqsubseteq$
 $\Delta_{\geq} f(x) \Leftrightarrow \forall x \in \mathbb{R} : \langle f \rangle \langle \iota_A|\mathbb{R}|_{\geq} \rangle^* \{x\} \sqsubseteq \langle \iota_B|\mathbb{R}|_{\geq} \rangle^* \langle f \rangle^* \{x\} \Leftrightarrow \forall x \in \mathbb{R} : f \circ \iota_A|\mathbb{R}|_{\geq} \sqsubseteq$
 $\iota_B|\mathbb{R}|_{\geq} \circ f \Leftrightarrow f \in C(\iota_A|\mathbb{R}|_{\geq}, \iota_B|\mathbb{R}|_{\geq})$.

Let $f \in C(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{>})$. Then $f \in C(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{\geq})$ and thus it is mono-
 tone. We need to prove that f is strictly monotone. Suppose the contrary. Then
 there is a nonempty interval $[p; q] \subseteq A$ such that f is constant on this interval. But
 this is impossible because $f \in C(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{>})$.

Prove that $f \in C(\iota_A|\mathbb{R}|_{\geq}, \iota_B|\mathbb{R}|_{\geq}) \cap C(\iota_A|\mathbb{R}|_{\leq}, \iota_B|\mathbb{R}|_{\leq})$ implies $f \in C(A, B)$.
 Really, it implies $\langle f \rangle \Delta_{\leq}(x) \sqsubseteq \Delta_{\leq}(f(x))$ and $\langle f \rangle \Delta_{\geq}(x) \sqsubseteq \Delta_{\geq}(f(x))$ thus $\langle f \rangle \Delta(x) =$
 $\langle f \rangle (\Delta_{\leq}(x) \sqcup \{x\} \sqcup \Delta_{\geq}(x)) \sqsubseteq \Delta_{\leq} f(x) \sqcup \{f(x)\} \sqcup \Delta_{\geq} f(x) = \Delta(f(x))$.

Prove that $f \in C(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{>}) \cap C(\iota_A|\mathbb{R}|_{<}, \iota_B|\mathbb{R}|_{<})$ implies $f \in C(A, B)$. Really, it
 implies $\langle f \rangle \Delta_{<}(x) \sqsubseteq \Delta_{<}(f(x))$ and $\langle f \rangle \Delta_{>}(x) \sqsubseteq \Delta_{>}(f(x))$ thus $\langle f \rangle \Delta(x) = \langle f \rangle (\Delta_{<}(x) \sqcup$
 $\{x\} \sqcup \Delta_{>}(x)) \sqsubseteq \Delta_{<} f(x) \sqcup \{f(x)\} \sqcup \Delta_{>} f(x) = \Delta(f(x))$. \square

THEOREM 1870. Let function $f : A \rightarrow B$ where $A, B \in \mathcal{P}\mathbb{R}$.

- 1°. f is locally monotone and $f \in C(A, B)$ iff $f \in C(A, B) \cap C(\iota_A|\mathbb{R}|_{\geq}, \iota_B|\mathbb{R}|_{\geq})$
 iff $f \in C(A, B) \cap C(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{\geq})$ iff $f \in C(\iota_A|\mathbb{R}|_{\geq}, \iota_B|\mathbb{R}|_{\geq}) \cap$
 $C(\iota_A|\mathbb{R}|_{\leq}, \iota_B|\mathbb{R}|_{\leq})$.
- 2°. f is locally strictly monotone and $\in C(A, B)$ iff $f \in C(A, B) \cap$
 $C(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{>})$ iff $f \in C(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{>}) \cap C(\iota_A|\mathbb{R}|_{<}, \iota_B|\mathbb{R}|_{<})$.

PROOF. By the lemma it is (strictly) monotone on each connected component. \square

See also related math.SE questions:

- 1°. <http://math.stackexchange.com/q/1473668/4876>
- 2°. <http://math.stackexchange.com/a/1872906/4876>
- 3°. <http://math.stackexchange.com/q/1875975/4876>

4.1. Directed topological spaces. Directed topological spaces are defined
 at

<http://ncatlab.org/nlab/show/directed+topological+space>

DEFINITION 1871. A *directed topological space* (or *d-space* for short) is a pair (X, d) of a topological space X and a set $d \subseteq C([0; 1], X)$ (called *directed paths* or *d-paths*) of paths in X such that

- 1°. (constant paths) every constant map $[0; 1] \rightarrow X$ is directed;
- 2°. (reparameterization) d is closed under composition with increasing continuous maps $[0; 1] \rightarrow [0; 1]$;
- 3°. (concatenation) d is closed under path-concatenation: if the d-paths a, b are consecutive in X ($a(1) = b(0)$), then their ordinary concatenation $a + b$ is also a d-path

$$(a + b)(t) = a(2t), \text{ if } 0 \leq t \leq \frac{1}{2},$$

$$(a + b)(t) = b(2t - 1), \text{ if } \frac{1}{2} \leq t \leq 1.$$

I propose a new way to construct a directed topological space. My way is more geometric/topological as it does not involve dealing with particular paths.

DEFINITION 1872. Let T be the complete endofunctor corresponding to a topological space and $\nu \sqsubseteq T$ be its “subfunctor”. The d-space $(\text{dir})(T, \nu)$ induced by the pair (T, ν) consists of T and paths $f \in C([0; 1], T) \cap C([0; 1]_{\geq}, \nu)$ such that $f(0) = f(1)$.

PROPOSITION 1873. It is really a d-space.

PROOF. Every d-path is continuous.

Constant path are d-paths because ν is reflexive.

Every reparameterization is a d-path because they are $C([0; 1]_{\geq}, \nu)$ and we can apply the theorem about composition of continuous functions.

Every concatenation is a d-path. Denote $f_0 = \lambda t \in [0; \frac{1}{2}] : a(2t)$ and $f_1 = \lambda t \in [\frac{1}{2}; 1] : b(2t - 1)$. Obviously $f_0, f_1 \in C([0; 1], \mu) \cap C([0; 1]_{\geq}, \nu)$. Then we conclude that $a + b = f_0 \sqcup f_1$ is in $f_0, f_1 \in C([0; 1], \mu) \cap C([0; 1]_{\geq}, \nu)$ using the fact that the operation \circ is distributive over \sqcup . \square

Below we show that not every d-space is induced by a pair of an endofunctor and its subfunctor. But are d-spaces not represented this way good anything except counterexamples?

Let now we have a d-space (X, d) . Define functor ν corresponding to the d-space by the formula $\nu = \bigsqcup_{a \in d} (a \circ |\mathbb{R}|_{\geq} \circ a^{-1})$.

EXAMPLE 1874. The two directed topological spaces, constructed from a fixed topological space and two different reflexive functors, are the same.

PROOF. Consider the indiscrete topology T on \mathbb{R} and the functors $1^{\text{FCD}(\mathbb{R}, \mathbb{R})}$ and $1^{\text{FCD}(\mathbb{R}, \mathbb{R})} \sqcup (\{0\} \times^{\text{FCD}} \Delta_{\geq})$. The only d-paths in both these settings are constant functions. \square

EXAMPLE 1875. A d-space is not determined by the induced functor.

PROOF. The following d-space induces the same functor as the d-space of all paths on the plane.

Consider a plane \mathbb{R}^2 with the usual topology. Let d-paths be paths lying inside a polygonal chain (in the plane). \square

CONJECTURE 1876. A d-path a is determined by the functors (where x spans $[0; 1]$)

$$(\lambda t \in \mathbb{R} : a(x + t))|_{\Delta(0)}.$$

5. A way to construct directed topological spaces

FiXme: Should include definition of directed topological space.

Directed topological spaces are defined at

<http://ncatlab.org/nlab/show/directed+topological+space>

I propose a new way to construct a directed topological space. My way is more geometric/topological as it does not involve dealing with particular paths.

CONJECTURE 1877. Every directed topological space can be constructed in the below described way.

Consider topological space T and its subfunctor F (that is F is a functor which is less than T in the order of functors). Note that in our consideration F is an endofunctor (its source and destination are the same).

Then a directed path from point A to point B is defined as a continuous function f from $[0; 1]$ to F such that $f(0) = A$ and $f(1) = B$. **FiXme:** Specify whether the interval $[0; 1]$ is treated as a proximity, pretopology, or preclosure.

Because F is less than T , we have that every directed path is a path.

CONJECTURE 1878. The two directed topological spaces, constructed from a fixed topological space and two different functors, are different.

For a counter-example of (which of the two?) the conjecture consider functor $T \sqcap (\mathbb{Q} \times^{\text{FCD}} \mathbb{Q})$ where T is the usual topology on real line. We need to consider stability of existence and uniqueness of a path under transformations of our functor and under transformations of the vector field. Can this be a step to solve Navier-Stokes existence and smoothness problems?

6. Integral curves

We will consider paths in a normed vector space V .

DEFINITION 1879. Let D be a connected subset of \mathbb{R} . A *path* is a function $D \rightarrow V$.

Let d be a vector field in a normed vector space V .

DEFINITION 1880. *Integral curve* of a vector field d is a differentiable function $f : D \rightarrow V$ such that $f'(t) = d(f(t))$ for every $t \in D$.

DEFINITION 1881. The definition of *right side integral curve* is the above definition with right derivative of f instead of derivative f' . *Left side integral curve* is defined similarly.

6.1. Path reparameterization. C^1 is a function which has continuous derivative on every point of the domain.

By D^1 I will denote a C^1 function whose derivative is either nonzero at every point or is zero everywhere.

DEFINITION 1882. A *reparameterization* of a C^1 path is a bijective C^1 function $\phi : D \rightarrow D$ such that $\phi'(t) > 0$. A curve f_2 is called a reparameterized curve f_1 if there is a reparameterization ϕ such that $f_2 = f_1 \circ \phi$.

It is well known that this defines an equivalence relation of functions.

PROPOSITION 1883. Reparameterization of D^1 function is D^1 .

PROOF. If the function has zero derivative, it is obvious.

Let f_1 has everywhere nonzero derivative. Then $f_2'(t) = f_1'(\phi(t))\phi'(t)$ what is trivially nonzero. \square

DEFINITION 1884. Vectors p and q have the *same direction* ($p \uparrow\uparrow q$) iff there exists a strictly positive real c such that $p = cq$.

OBVIOUS 1885. Being same direction is an equivalence relation.

OBVIOUS 1886. The only vector with the same direction as the zero vector is zero vector.

THEOREM 1887. A D^1 function y is some reparameterization of a D^1 integral curve x of a continuous vector field d iff $y'(t) \uparrow\uparrow d(y(t))$ that is vectors $y'(t)$ and $d(y(t))$ have the same direction (for every t).

PROOF. If y is a reparameterization of x , then $y(t) = x(\phi(t))$. Thus $y'(t) = x'(\phi(t))\phi'(t) = d(x(\phi(t)))\phi'(t) = d(y(t))\phi'(t)$. So $y'(t) \uparrow\uparrow d(y(t))$ because $\phi'(t) > 0$.

Let now $x'(t) \uparrow\uparrow d(x(t))$ that is that is there is a strictly positive function $c(t)$ such that $x'(t) = c(t)d(x(t))$.

If $x'(t)$ is zero everywhere, then $d(x(t)) = 0$ and thus $x'(t) = d(x(t))$ that is x is an Integral curve. Note that y is a reparameterization of itself.

We can assume that $x'(t) \neq 0$ everywhere. Then $F(x(t)) \neq 0$. We have that $c(t) = \frac{\|x'(t)\|}{\|d(x(t))\|}$ is a continuous function. **FiXme: Does it work for non-normed spaces?**

Let $y(u(t)) = x(t)$, where

$$u(t) = \int_0^t c(s)ds,$$

which is defined and finite because c is continuous and monotone (thus having inverse defined on its image) because c is positive.

Then

$$\begin{aligned} y'(u(t))u'(t) &= x'(t) \\ &= c(t)d(x(t)), \text{ so} \\ y'(u(t))c(t) &= c(t)d(y(u(t))) \\ y'(u(t)) &= d(y(u(t))) \end{aligned}$$

and letting $s = u(t)$ we have $y'(s) = d(y(s))$ for a reparameterization y of x . \square

6.2. Vector space with additional coordinate. Consider the normed vector space with additional coordinate t .

Our vector field $d(t)$ induces vector field $\hat{d}(t, v) = (1, d(v))$ in this space. Also $\hat{f}(t) = (t, f(t))$.

PROPOSITION 1888. Let f be a D^1 function. f is an integral curve of d iff \hat{f} is a reparametrized integral curve of \hat{d} .

PROOF. First note that \hat{f} always has a nonzero derivative. $\hat{f}'(t) \uparrow\uparrow \hat{d}(\hat{f}(t)) \Leftrightarrow (1, f'(t)) \uparrow\uparrow (1, d(f(t))) \Leftrightarrow f'(t) = d(f(t))$. \square

Thus we have reduced (for D^1 paths) being an integral curve to being a reparametrized integral curve. We will also describe being a reparametrized integral curve topologically (through funcoids).

6.3. Topological description of C^1 curves. Explicitly construct this funcoid as follows:

$R(d, \phi) = \left\{ \frac{v \in V}{v \hat{d} < \phi, v \neq 0} \right\}$ for $d \neq 0$ and $R(0, \phi) = \{0\}$, where $\hat{a}b$ is the angle between the vectors a and b , for a direction d and an angle ϕ .

DEFINITION 1889. $W(d) = \prod^{\text{RLD}} \left\{ \frac{R(d, \phi)}{\phi \in \mathbb{R}, \phi > 0} \right\} \cap \prod_{r>0}^{\text{RLD}} B_r(0)$. **FixMe:** This is defined for infinite dimensional case. **FixMe:** Consider also FCD instead of RLD.

PROPOSITION 1890. For finite dimensional case \mathbb{R}^n we have $W(d) = \prod^{\text{RLD}} \left\{ \frac{R(d, \phi)}{\phi \in \mathbb{R}, \phi > 0} \right\} \cap \Delta^{(\text{RLD})n}$ where

$$\Delta^{(\text{RLD})n} = \underbrace{\Delta \times^{\text{RLD}} \dots \times^{\text{RLD}} \Delta}_{n \text{ times}}.$$

PROOF. ?? □

Finally our funcoinds are the complete funcoinds Q_+ and Q_- described by the formulas

$$\langle Q_+ \rangle^* @ \{p\} = \langle p+ \rangle W(d(p)) \quad \text{and} \quad \langle Q_- \rangle^* @ \{p\} = \langle p+ \rangle W(-d(p)).$$

Here Δ is taken from the “counter-examples” section.

In other words,

$$Q_+ = \bigsqcup_{p \in \mathbb{R}} (\@ \{p\} \times^{\text{FCD}} \langle p+ \rangle W(d(p))); \quad Q_- = \bigsqcup_{p \in \mathbb{R}} (\@ \{p\} \times^{\text{FCD}} \langle p+ \rangle W(-d(p))).$$

That is $\langle Q_+ \rangle^* @ \{p\}$ and $\langle Q_- \rangle^* @ \{p\}$ are something like infinitely small spherical sectors (with infinitely small aperture and infinitely small radius).

FixMe: Describe the co-complete funcoinds reverse to these complete funcoinds.

THEOREM 1891. A D^1 curve f is an reparametrized integral curve for a direction field d iff $f \in C(\iota_D | \mathbb{R}|_>, Q_+) \cap C(\iota_D | \mathbb{R}|_<, Q_-)$.

PROOF. Equivalently transform $f \in C(\iota_D | \mathbb{R}|, Q_+)$; $f \circ \iota_D | \mathbb{R}| \sqsubseteq Q_+ \circ f$; $\langle f \circ \iota_D | \mathbb{R}| \rangle^* @ \{t\} \sqsubseteq \langle Q_+ \circ f \rangle^* @ \{t\}$; $\langle f \rangle^* \Delta_>(t) \cap D \sqsubseteq \langle Q_+ \rangle^* f(t)$; $\langle f \rangle^* \Delta_>(t) \sqsubseteq \langle Q_+ \rangle^* f(t)$; $\langle f \rangle^* \Delta_>(t) \sqsubseteq f(t) + W(D(f(t)))$; $\langle f \rangle^* \Delta_>(t) - f(t) \sqsubseteq W(D(f(t)))$;

$$\forall r > 0, \phi > 0 \exists \delta > 0 : \langle f \rangle^* (]t; t + \delta]) - f(t) \subseteq R(d(f(t)), \phi) \cap B_r(f(t));$$

$$\forall r > 0, \phi > 0 \exists \delta > 0 \forall 0 < \gamma < \delta : \langle f \rangle^* (]t; t + \gamma]) - f(t) \subseteq R(d(f(t)), \phi) \cap B_r(f(t));$$

$$\forall r > 0, \phi > 0 \exists \delta > 0 \forall 0 < \gamma < \delta : \frac{\langle f \rangle^* (]t; t + \gamma]) - f(t)}{\gamma} \subseteq R(d(f(t)), \phi) \cap B_{r/\delta}(f(t));$$

$$\forall r > 0, \phi > 0 \exists \delta > 0 : \partial_+ f(t) \subseteq R(d(f(t)), \phi) \cap B_{r/\delta}(f(t));$$

$$\forall r > 0, \phi > 0 : \partial_+ f(t) \subseteq R(d(f(t)), \phi);$$

$$\partial_+ f(t) \uparrow\uparrow d(f(t))$$

where ∂_+ is the right derivative.

In the same way we derive that $C(|\mathbb{R}|_<, Q_-) \Leftrightarrow \partial_- f(t) \uparrow\uparrow d(f(t))$.

Thus $f'(t) \uparrow\uparrow d(f(t))$ iff $f \in C(|\mathbb{R}|_>, Q_+) \cap C(|\mathbb{R}|_<, Q_-)$. □

The following idea seems wrong. I grayed it out as a candidate for deletion from the text:

6.4. C^n curves. **FixMe:** Related questions: <http://math.stackexchange.com/q/1884856/4876> <http://math.stackexchange.com/q/107460/4876> <http://mathoverflow.net/q/88501>

$$\text{Define } R^n(d) = \left\{ \frac{v \in V}{vd < o(|v|^n), v \neq 0} \right\} \text{ for } d \neq 0 \text{ and } R^n(0) = \{0\}.$$

DEFINITION 1892. $W^n(d) = R^n(d) \cap \prod_{r>0}^{\text{RLD}} B_r(0)$.

Finally our funcoinds are the complete funcoinds Q_+^n and Q_-^n described by the formulas

$$\langle Q_+^n \rangle^* @ \{p\} = \langle p+ \rangle W^n(d(p)) \quad \text{and} \quad \langle Q_-^n \rangle^* @ \{p\} = \langle p+ \rangle W^n(-d(p)).$$

LEMMA 1893. Let for every x in the domain of the path for small $t > 0$ we have $f(x+t) \in R^n(F(f(x)))$ and $f(x-t) \in R^n(-F(f(x)))$. Then f is C^n smooth.

PROOF. **FiXme: Not yet proved!**

See also <http://math.stackexchange.com/q/1884930/4876>. \square

CONJECTURE 1894. A path f is C^n smooth iff $f \in C(\iota_D|\mathbb{R}|_{>}, Q_+^n) \cap C(\iota_D|\mathbb{R}|_{<}, Q_-^n)$.

PROOF. Reverse implication follows from the lemma.

Let now a path f is C^n . Then

$$f(x+t) = \sum_{i=0}^n f^{(i)}(x) \frac{t^i}{i!} + o(t^i) = f(x) + f'(x)t + \sum_{i=2}^n f^{(i)}(x) \frac{t^i}{i!} + o(t^i)$$

\square

6.5. Plural funcoids. Take I_+ and Q_+ as described above in forward direction and I_- and Q_- in backward direction. Then

$$f \in C(I_+, Q_+) \wedge f \in C(I_-, Q_-) \Leftrightarrow f \times f \in C(I_+ \times^{(A)} I_-, Q_+ \times^{(A)} Q_-)?$$

To describe the above we can introduce new term *plural funcoids*. This is simply a map from an index set to funcoids. Composition is defined component-wise. Order is defined as product order. Well, do we need this? Isn't it the same as infimum product of funcoids?

6.6. Multiple allowed directions per point.

$$\langle Q \rangle^* @ \{p\} = \bigsqcup_{d \in d(p)} \langle p+ \rangle W(d).$$

It seems (check!) that solutions not only of differential equations but also of difference equations can be expressed as paths in funcoids.

CHAPTER 3

Generalized cofinite filters

The following is a straightforward generalization of cofinite filter on a coatomic poset.

DEFINITION 1895. $\Omega_{1a} = \prod_{X \in \text{coatoms}^{\mathfrak{A}}} X$; $\Omega_{1b} = \prod_{X \in \text{coatoms}^{\mathfrak{F}}} X$.

PROPOSITION 1896. For primary filtrators $\Omega_{1a} = \Omega_{1b}$.

PROOF. Proposition 531. □

Thus for primary filtrators I will denote it just Ω .

PROPOSITION 1897. Let \mathfrak{A} be a subset of $\mathcal{P}U$. Let it be a meet-semilattice with greatest element **Fixme: existence of greatest element seems unnecessary**. Let also every non-coempty cofinite set lies in \mathfrak{A} . Then

$$\partial\Omega = \left\{ \frac{Y \in \mathfrak{A}}{\text{card atoms}^{\mathfrak{A}} Y \geq \omega} \right\}. \quad (1)$$

PROOF. Ω exists by corollary 496.

$Y \in \partial\Omega \Leftrightarrow Y \not\leq \prod_{X \in \text{coatoms}^{\mathfrak{A}}} X \Leftrightarrow$ (by properties of filter bases) $\Leftrightarrow \forall S \in \mathcal{P}_{\text{fin}} \text{coatoms}^{\mathfrak{A}} : Y \not\leq \prod_{S \in \mathcal{P}_{\text{fin}} \text{coatoms}^{\mathfrak{A}}} S \Leftrightarrow$ (theorem 512) $\Leftrightarrow \forall S \in \mathcal{P}_{\text{fin}} \text{coatoms}^{\mathfrak{A}} : Y \not\leq \prod S \Leftrightarrow \forall K \in \mathcal{P}_{\text{fin}} U : Y \setminus K \neq \emptyset \Leftrightarrow \text{card } Y \geq \omega \Leftrightarrow \text{card atoms}^{\mathfrak{A}} Y \geq \omega$. **Fixme:** Define \mathcal{P}_{fin} . □

COROLLARY 1898. Formula (1) holds for both reloids and functors.

PROOF. For reloids it's straightforward, for functors take that they are isomorphic to filters on lattice Γ . □

COROLLARY 1899. $\Omega^{\text{FCD}} \neq \perp^{\text{FCD}}$ (for $\text{FCD}(A, B)$ where $A \times B$ is an infinite set).

PROPOSITION 1900. $\text{up } \Omega = \left\{ \frac{\prod S}{S \in \mathcal{P}_{\text{fin}} \text{coatoms}^{\mathfrak{A}}} \right\}$.

PROOF. Because $\left\{ \frac{\prod S}{S \in \mathcal{P}_{\text{fin}} \text{coatoms}^{\mathfrak{A}}} \right\}$ is a filter. □

COROLLARY 1901. $\text{up } \Omega^{\text{FCD}} = \text{up } \Omega^{\text{RLD}}$.

PROPOSITION 1902.

1°. $\langle \Omega^{\text{FCD}} \rangle \{x\} = \Omega^U$;

2°. $\langle \Omega^{\text{FCD}} \rangle p = \top$ for every nontrivial atomic filter p .

PROOF. $\langle \Omega^{\text{FCD}} \rangle \{x\} = \prod_{y \in U} (U \setminus \{y\}) = \Omega^U$; $\langle \Omega^{\text{FCD}} \rangle p = \prod_{y \in U} \top = \top$. □

PROPOSITION 1903. $(\text{FCD})\Omega^{\text{RLD}} = \Omega^{\text{FCD}}$.

PROOF. $(\text{FCD})\Omega^{\text{RLD}} = \prod^{\text{FCD}} \text{up } \Omega^{\text{RLD}} = \Omega^{\text{FCD}}$. □

PROPOSITION 1904. $(\text{RLD})_{\text{out}} \Omega^{\text{FCD}} = \Omega^{\text{RLD}}$.

PROOF. $(\text{RLD})_{\text{out}} \Omega^{\text{FCD}} = \prod^{\text{RLD}} \text{up } \Omega^{\text{FCD}} = \prod^{\text{RLD}} \text{up } \Omega^{\text{RLD}} = \Omega^{\text{RLD}}$. □

PROPOSITION 1905. $(\text{RLD})_{\text{in}\Omega^{\text{FCD}}} = \Omega^{\text{RLD}}$.

PROOF.

$$\begin{aligned}
(\text{RLD})_{\text{in}\Omega^{\text{FCD}}} &= \bigsqcup \left\{ \frac{a \times^{\text{RLD}} b}{a \in \text{atoms}^{\mathcal{F}}, b \in \text{atoms}^{\mathcal{F}}, a \times^{\text{FCD}} b \sqsubseteq \Omega^{\text{FCD}}} \right\} = \\
&\bigsqcup \left\{ \frac{a \times^{\text{RLD}} b}{a \in \text{atoms}^{\mathcal{F}}, b \in \text{atoms}^{\mathcal{F}}, \text{not } a \text{ and } b \text{ both atomic}} \right\} = \\
&\bigsqcup \left\{ \frac{\bigsqcup \text{atoms}(a \times^{\text{RLD}} b)}{a \in \text{atoms}^{\mathcal{F}}, b \in \text{atoms}^{\mathcal{F}}, \text{not } a \text{ and } b \text{ both atomic}} \right\} = \\
&\bigsqcup \bigsqcup \left\{ \frac{\text{atoms}(a \times^{\text{RLD}} b)}{a \in \text{atoms}^{\mathcal{F}}, b \in \text{atoms}^{\mathcal{F}}, \text{not } a \text{ and } b \text{ both atomic}} \right\} = \\
&\bigsqcup (\text{nontrivial atomic reloids under } A \times B) = \Omega^{\text{RLD}}.
\end{aligned}$$

□

Extending Galois connections between funcoids and reloids

DEFINITION 1906.

- 1°. $\Phi_* f = \lambda b \in \mathfrak{B} : \sqcup \left\{ \frac{x \in \mathfrak{A}}{fx \sqsubseteq b} \right\};$
 2°. $\Phi^* f = \lambda b \in \mathfrak{A} : \prod \left\{ \frac{x \in \mathfrak{B}}{fx \sqsupseteq b} \right\}.$

PROPOSITION 1907.

- 1°. If f has upper adjoint then $\Phi_* f$ is the upper adjoint of f .
 2°. If f has lower adjoint then $\Phi^* f$ is the lower adjoint of f .

PROOF. By theorem 126. □

LEMMA 1908. $\Phi^*(\text{RLD})_{\text{out}} = (\text{FCD}).$

PROOF. $(\Phi^*(\text{RLD})_{\text{out}})f = \prod \left\{ \frac{g \in \text{FCD}}{(\text{RLD})_{\text{out}} g \sqsupseteq f} \right\} = \prod^{\text{FCD}} \left\{ \frac{g \in \mathbf{Rel}}{(\text{RLD})_{\text{out}} g \sqsupseteq f} \right\} =$
 $\prod^{\text{FCD}} \left\{ \frac{g \in \mathbf{Rel}}{g \sqsupseteq f} \right\} = (\text{FCD})f.$ □

LEMMA 1909. $\Phi_*(\text{RLD})_{\text{out}} \neq (\text{FCD}).$

PROOF. $(\Phi_*(\text{RLD})_{\text{out}})f = \sqcup \left\{ \frac{g \in \text{FCD}}{(\text{RLD})_{\text{out}} g \sqsubseteq f} \right\}$
 $(\Phi_*(\text{RLD})_{\text{out}}) \perp \neq \perp.$ □

LEMMA 1910. $\Phi^*(\text{FCD}) = (\text{RLD})_{\text{out}}.$

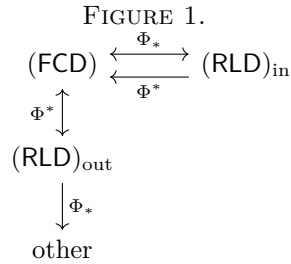
PROOF. $(\Phi^*(\text{FCD}))f = \prod \left\{ \frac{g \in \text{RLD}}{(\text{FCD})g \sqsupseteq f} \right\} = \prod^{\text{RLD}} \left\{ \frac{g \in \mathbf{Rel}}{(\text{FCD})g \sqsupseteq f} \right\} = \prod^{\text{RLD}} \left\{ \frac{g \in \mathbf{Rel}}{g \sqsupseteq f} \right\} =$
 $(\text{RLD})_{\text{out}}f.$ □

LEMMA 1911. $\Phi_*(\text{RLD})_{\text{in}} = (\text{FCD}).$

PROOF. $(\Phi_*(\text{RLD})_{\text{in}})f = \sqcup \left\{ \frac{g \in \text{FCD}}{(\text{RLD})_{\text{in}} g \sqsubseteq f} \right\} = \sqcup \left\{ \frac{g \in \text{FCD}}{g \sqsubseteq (\text{FCD})f} \right\} = (\text{FCD})f.$ □

THEOREM 1912. The picture at figure 1 describes values of functions Φ_* and Φ^* . All nodes of this diagram are distinct.

PROOF. Follows from the above lemmas. □



QUESTION 1913. What happens if we keep applying Φ^* and Φ_* to the node “other”? Will we this way get a finite or infinite set?

Boolean funcoids

1. One-element boolean lattice

Let \mathfrak{A} be a boolean lattice and $\mathfrak{B} = \mathcal{P}0$. It's sole element is \perp .

$$f \in \text{pFCD}(\mathfrak{A}; \mathfrak{B}) \Leftrightarrow \forall X \in \mathfrak{A} : (\langle f \rangle X \neq \perp \Leftrightarrow \langle f^{-1} \rangle \perp \neq X) \Leftrightarrow \forall X \in \mathfrak{A} : (0 \Leftrightarrow \langle f^{-1} \rangle \perp \neq X) \Leftrightarrow \forall X \in \mathfrak{A} : \langle f^{-1} \rangle \perp \simeq X \Leftrightarrow \forall X \in \mathfrak{A} : \langle f^{-1} \rangle \perp = \perp^{\mathfrak{A}} \Leftrightarrow \langle f^{-1} \rangle \perp = \perp^{\mathfrak{A}} \Leftrightarrow \langle f^{-1} \rangle = \{(\perp; \perp^{\mathfrak{A}})\}.$$

Thus $\text{card pFCD}(\mathfrak{A}; \mathcal{P}0) = 1$.

2. Two-element boolean lattice

Consider the two-element boolean lattice $\mathfrak{B} = \mathcal{P}1$.

Let f be a pointfree protofuncoid from \mathfrak{A} to \mathfrak{B} (that is $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)$ where $\alpha \in \mathfrak{B}^{\mathfrak{A}}, \beta \in \mathfrak{A}^{\mathfrak{B}}$).

$$f \in \text{pFCD}(\mathfrak{A}; \mathfrak{B}) \Leftrightarrow \forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\langle f \rangle X \neq Y \Leftrightarrow \langle f^{-1} \rangle Y \neq X) \Leftrightarrow \forall X \in \mathfrak{A}, Y \in \mathfrak{B} : ((0 \in \langle f \rangle X \wedge 0 \in Y) \vee (1 \in \langle f \rangle X \wedge 1 \in Y) \Leftrightarrow \langle f^{-1} \rangle Y \neq X).$$

$T = \left\{ \frac{X \in \mathfrak{A}}{0 \in \langle f \rangle X} \right\}$ is an ideal. Really: That it's an upper set is obvious. Let $P \cup Q \in \left\{ \frac{X \in \mathfrak{A}}{0 \in \langle f \rangle X} \right\}$. Then $0 \in \langle f \rangle (P \cup Q) = \langle f \rangle P \cup \langle f \rangle Q$; $0 \in \langle f \rangle P \vee 0 \in \langle f \rangle Q$.

Similarly $S = \left\{ \frac{X \in \mathfrak{A}}{1 \in \langle f \rangle X} \right\}$ is an ideal.

Let now $T, S \in \mathcal{P}\mathfrak{A}$ be ideals. Can we restore $\langle f \rangle$? Yes, because we know $0 \in \langle f \rangle X$ and $1 \in \langle f \rangle X$ for every $X \in \mathfrak{A}$.

So it is equivalent to $\forall X \in \mathfrak{A}, Y \in \mathfrak{B} : ((X \in T \wedge 0 \in Y) \vee (X \in S \wedge 1 \in Y) \Leftrightarrow \langle f^{-1} \rangle Y \neq X)$.

$f \in \text{pFCD}(\mathfrak{A}; \mathfrak{B})$ is equivalent to conjunction of all rows of this table:

Y	equality
\emptyset	$\langle f^{-1} \rangle \emptyset = \emptyset$
$\{0\}$	$X \in T \Leftrightarrow \langle f^{-1} \rangle \{0\} \neq X$
$\{1\}$	$X \in S \Leftrightarrow \langle f^{-1} \rangle \{1\} \neq X$
$\{0,1\}$	$X \in T \vee X \in S \Leftrightarrow \langle f^{-1} \rangle \{0,1\} \neq X$

Simplified:

Y	equality
\emptyset	$\langle f^{-1} \rangle \emptyset = \emptyset$
$\{0\}$	$T = \partial \langle f^{-1} \rangle \{0\}$
$\{1\}$	$S = \partial \langle f^{-1} \rangle \{1\}$
$\{0,1\}$	$T \cup S = \partial \langle f^{-1} \rangle \{0,1\}$

From the last table it follows that T and S are principal ideals.

So we can take arbitrary either $\langle f^{-1} \rangle \{0\}$, $\langle f^{-1} \rangle \{1\}$ or principal ideals T and S .

In other words, we take $\langle f^{-1} \rangle \{0\}$, $\langle f^{-1} \rangle \{1\}$ arbitrary and independently. So we have $\text{pFCD}(\mathfrak{A}; \mathfrak{B})$ equivalent to product of two instances of \mathfrak{A} . So it a boolean lattice. **FiXme: I messed product with disjoint union below.)**

3. Finite boolean lattices

We can assume $\mathfrak{B} = \mathcal{P}B$ for a set B , $\text{card } B = n$. Then

$$f \in \text{pFCD}(\mathfrak{A}; \mathfrak{B}) \Leftrightarrow \forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\langle f \rangle X \neq Y \Leftrightarrow \langle f^{-1} \rangle Y \neq X) \Leftrightarrow \forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\exists i \in Y : i \in \langle f \rangle X \Leftrightarrow \langle f^{-1} \rangle Y \neq X).$$

Having values of $\langle f^{-1} \rangle \{i\}$ we can restore all $\langle f^{-1} \rangle Y$. [need this paragraph?]

$$\text{Let } T_i = \left\{ \frac{X \in \mathfrak{A}}{i \in \langle f \rangle X} \right\}.$$

Let now $T_i \in \mathcal{P}\mathfrak{A}$ be ideals. Can we restore $\langle f \rangle$? Yes, because we know $i \in \langle f \rangle X$ for every $X \in \mathfrak{A}$.

So, it is equivalent to:

$$\forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\exists i \in Y : X \in T_i \Leftrightarrow \langle f^{-1} \rangle Y \neq X). \quad (2)$$

LEMMA 1914. The formula (2) is equivalent to:

$$\forall X \in \mathfrak{A}, i \in B : (X \in T_i \Leftrightarrow \langle f^{-1} \rangle \{i\} \neq X). \quad (3)$$

PROOF. (2) \Rightarrow (3). Just take $Y = \{i\}$.

(3) \Rightarrow (2). Let (3) holds. Let also $X \in \mathfrak{A}, Y \in \mathfrak{B}$. Then $\langle f^{-1} \rangle Y \neq X \Leftrightarrow \bigcup_{i \in Y} \langle f^{-1} \rangle \{i\} \neq X \Leftrightarrow \exists i \in Y : \langle f^{-1} \rangle \{i\} \neq X \Leftrightarrow \exists i \in Y : X \in T_i$. \square

Further transforming: $\forall i \in B : T_i = \partial \langle f^{-1} \rangle \{i\}$.

So $\langle f^{-1} \rangle \{i\}$ are arbitrary elements of \mathfrak{B} and T_i are corresponding arbitrary principal ideals.

In other words, $\text{pFCD}(\mathfrak{A}; \mathfrak{B}) \cong \mathfrak{A}\Pi \dots \Pi \mathfrak{A}$ ($\text{card } B$ times). Thus $\text{pFCD}(\mathfrak{A}; \mathfrak{B})$ is a boolean lattice.

4. About infinite case

Let \mathfrak{A} be a complete boolean lattice, \mathfrak{B} be an atomistic boolean lattice.

$$f \in \text{pFCD}(\mathfrak{A}; \mathfrak{B}) \Leftrightarrow \forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\langle f \rangle X \neq Y \Leftrightarrow \langle f^{-1} \rangle Y \neq X) \Leftrightarrow \forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\exists i \in \text{atoms } Y : i \in \text{atoms } \langle f \rangle X \Leftrightarrow \langle f^{-1} \rangle Y \neq X).$$

$$\text{Let } T_i = \left\{ \frac{X \in \mathfrak{A}}{i \in \text{atoms } \langle f \rangle X} \right\}.$$

T_i is an ideal: Really: That it's an upper set is obvious. Let $P \cup Q \in \left\{ \frac{X \in \mathfrak{A}}{i \in \text{atoms } \langle f \rangle X} \right\}$. Then $i \in \text{atoms } \langle f \rangle (P \cup Q) = \text{atoms } \langle f \rangle P \cup \text{atoms } \langle f \rangle Q; i \in \langle f \rangle P \vee i \in \langle f \rangle Q$.

Let now $T_i \in \mathcal{P}\mathfrak{A}$ be ideals. Can we restore $\langle f \rangle$? Yes, because we know $i \in \text{atoms } \langle f \rangle X$ for every $X \in \mathfrak{A}$ and \mathfrak{B} is atomistic.

So, it is equivalent to:

$$\forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\exists i \in \text{atoms } Y : X \in T_i \Leftrightarrow \langle f^{-1} \rangle Y \neq X). \quad (4)$$

LEMMA 1915. The formula (4) is equivalent to:

$$\forall X \in \mathfrak{A}, i \in \text{atoms } \mathfrak{B} : (X \in T_i \Leftrightarrow \langle f^{-1} \rangle i \neq X). \quad (5)$$

PROOF. (4) \Rightarrow (5). Let (4) holds. Take $Y = i$. Then $\text{atoms } Y = \{i\}$ and thus $X \in T_i \Leftrightarrow \exists i \in \text{atoms } Y : X \in T_i \Leftrightarrow \langle f^{-1} \rangle Y \neq X \Leftrightarrow \langle f^{-1} \rangle i \neq X$.

(5) \Rightarrow (4). Let (5) holds. Let also $X \in \mathfrak{A}, Y \in \mathfrak{B}$. Then $\langle f^{-1} \rangle Y \neq X \Leftrightarrow \langle f^{-1} \rangle \bigsqcup \text{atoms } Y \neq X \Leftrightarrow \bigsqcup_{i \in \text{atoms } Y} \langle f^{-1} \rangle i \neq X \Leftrightarrow \exists i \in \text{atoms } Y : \langle f^{-1} \rangle i \neq X \Leftrightarrow \exists i \in \text{atoms } Y : X \in T_i$. \square

Further equivalently transforming: $\forall i \in \text{atoms } \mathfrak{B} : T_i = \partial \langle f^{-1} \rangle i$.

So $\langle f^{-1} \rangle i$ are arbitrary elements of \mathfrak{B} and T_i are corresponding arbitrary principal ideals.

In other words, $\text{pFCD}(\mathfrak{A}; \mathfrak{B}) \cong \prod_{i \in \text{card atoms } \mathfrak{B}} \mathfrak{A}$. Thus $\text{pFCD}(\mathfrak{A}; \mathfrak{B})$ is a boolean lattice.

So finally we have a very weird theorem, which is a partial solution for the above open problem (The weirdness is in its partiality and asymmetry):

THEOREM 1916. If \mathfrak{A} is a complete boolean lattice and \mathfrak{B} is an atomistic boolean lattice (or vice versa), then $\text{pFCD}(\mathfrak{A}; \mathfrak{B})$ is a boolean lattice.

[4] proves “**THEOREM 4.6.** Let A, B be bounded posets. $A \otimes B$ is a completely distributive complete Boolean lattice iff A and B are completely distributive Boolean lattices.” (where $A \otimes B$ is equivalent to the set of Galois connections between A and B) and other interesting results.

Interior funcoids

Having a funcoid f let define *interior funcoid* f° .

DEFINITION 1917. Let $f \in \text{FCD}(A, B) = \text{pFCD}(\mathcal{T}A, \mathcal{T}B)$ be a co-complete funcoid. Then $f^\circ \in \text{pFCD}(\text{dual } \mathcal{T}A, \text{dual } \mathcal{T}B)$ is defined by the formula $\langle f^\circ \rangle^* X = \overline{\langle f \rangle X}$.

PROPOSITION 1918. Pointfree funcoid f° exists and is unique.

PROOF. $X \mapsto \overline{\langle f \rangle X}$ is a component of pointfree funcoid $\text{dual } \mathcal{T}A \rightarrow \text{dual } \mathcal{T}B$ iff $\langle f \rangle$ is a component of the corresponding pointfree funcoid $\mathcal{T}A \rightarrow \mathcal{T}B$ that is essentially component of the corresponding funcoid $\text{FCD}(A, B)$ what holds for a unique funcoid. \square

It can be also defined for arbitrary funcoids by the formula $f^\circ = (\text{CoCompl } f)^\circ$.

OBVIOUS 1919. f° is co-complete.

THEOREM 1920. The following values are pairwise equal for a co-complete funcoid f and $X \in \mathcal{T} \text{Src } f$:

- 1 $^\circ$. $\langle f^\circ \rangle^* X$;
- 2 $^\circ$. $\left\{ \frac{y \in \text{Dst } f}{\langle f^{-1} \rangle^* \{y\} \sqsubseteq X} \right\}$
- 3 $^\circ$. $\bigsqcup \left\{ \frac{Y \in \mathcal{T} \text{Dst } f}{\langle f^{-1} \rangle^* Y \sqsubseteq X} \right\}$
- 4 $^\circ$. $\bigsqcup \left\{ \frac{\mathcal{Y} \in \mathcal{F} \text{Dst } f}{\langle f^{-1} \rangle \mathcal{Y} \sqsubseteq X} \right\}$

PROOF.

$$1^\circ = 2^\circ. \left\{ \frac{y \in \text{Dst } f}{\langle f^{-1} \rangle^* \{y\} \sqsubseteq X} \right\} = \left\{ \frac{x \in \text{Dst } f}{\langle f^{-1} \rangle^* \{x\} \succ \overline{X}} \right\} = \left\{ \frac{x \in \text{Dst } f}{\{x\} \succ \langle f \rangle \overline{X}} \right\} = \overline{\langle f \rangle \overline{X}} = \langle f^\circ \rangle^* X.$$

2 $^\circ$ = 3 $^\circ$. If $\langle f^{-1} \rangle^* Y \sqsubseteq X$ then (by completeness of f^{-1}) $Y = \left\{ \frac{y \in Y}{\langle f^{-1} \rangle^* \{y\} \sqsubseteq X} \right\}$ and thus

$$\bigsqcup \left\{ \frac{Y \in \mathcal{T} \text{Dst } f}{\langle f^{-1} \rangle^* Y \sqsubseteq X} \right\} \sqsubseteq \left\{ \frac{y \in \text{Dst } f}{\langle f^{-1} \rangle^* \{y\} \sqsubseteq X} \right\}.$$

The reverse inequality is obvious.

3 $^\circ$ = 4 $^\circ$. It's enough to prove that if $\langle f^{-1} \rangle \mathcal{Y} \sqsubseteq X$ for $\mathcal{Y} \in \mathcal{F} \text{Dst } f$ then exists $Y \in \text{up } \mathcal{Y}$ such that $\langle f^{-1} \rangle^* Y \sqsubseteq X$. Really let $\langle f^{-1} \rangle \mathcal{Y} \sqsubseteq X$. Then $\bigsqcap \langle \langle f^{-1} \rangle^* \rangle \text{up } \mathcal{Y} \sqsubseteq X$ and thus exists $Y \in \text{up } \mathcal{Y}$ such that $\langle f^{-1} \rangle^* Y \sqsubseteq X$ by properties of generalized filter bases. \square

This coincides with the customary definition of interior in topological spaces.

PROPOSITION 1921. $f^{\circ\circ} = f$ for every funcoid f .

PROOF. $\langle f^{\circ\circ} \rangle^* X = \neg \neg \langle f \rangle \neg \neg X = \langle f \rangle X$. \square

PROPOSITION 1922. Let $g \in \text{FCD}(A, B)$, $f \in \text{FCD}(B, C)$, $h \in \text{FCD}(A, C)$ for some sets A, B, C .

$g \sqsubseteq f^\circ \circ h \Leftrightarrow f^{-1} \circ g \sqsubseteq h$, provided f and h are co-complete.

PROOF. $g \sqsubseteq f^\circ \circ h \Leftrightarrow \forall X \in A : \langle g \rangle^* X \sqsubseteq \langle f^\circ \circ h \rangle^* X \Leftrightarrow \forall X \in A : \langle g \rangle^* X \sqsubseteq \langle f^\circ \rangle^* \langle h \rangle^* X \Leftrightarrow \forall X \in A : \langle g \rangle^* X \sqsubseteq \neg \langle f \rangle^* \neg \langle h \rangle^* X \Leftrightarrow \forall X \in A : \langle g \rangle^* X \simeq \langle f \rangle^* \neg \langle h \rangle^* X \Leftrightarrow \forall X \in A : \langle f^{-1} \rangle^* \langle g \rangle^* X \simeq \neg \langle h \rangle^* X \Leftrightarrow \forall X \in A : \langle f^{-1} \rangle^* \langle g \rangle^* X \sqsubseteq \langle h \rangle^* X \Leftrightarrow \forall X \in A : \langle f^{-1} \circ g \rangle^* X \sqsubseteq \langle h \rangle^* X \Leftrightarrow f^{-1} \circ g \sqsubseteq h. \quad \square$

REMARK 1923. The above theorem allows to get rid of interior functors (and use only “regular” functors) in some formulas.

Filterization of pointfree funcoids

Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. By corollary 497 we have that \mathfrak{A} and \mathfrak{B} are complete lattices.

Let f be a pointfree funcoid $\mathfrak{Z}_0 \rightarrow \mathfrak{Z}_1$. Define pointfree funcoid $\uparrow f$ (*filterization* of f) by the formulas

$$\langle \uparrow f \rangle \mathcal{X} = \prod_{X \in \text{up } \mathcal{X}}^{\mathfrak{B}} \langle f \rangle X \quad \text{and} \quad \langle \uparrow f^{-1} \rangle \mathcal{Y} = \prod_{Y \in \text{up } \mathcal{Y}}^{\mathfrak{A}} \langle f^{-1} \rangle Y.$$

PROPOSITION 1924. $\uparrow f$ is a pointfree funcoid.

PROOF.

$$\begin{aligned} \mathcal{Y} \neq \langle \uparrow f \rangle \mathcal{X} &\Leftrightarrow \mathcal{Y} \neq \prod_{X \in \text{up } \mathcal{X}}^{\mathfrak{B}} \langle f \rangle X \Leftrightarrow \\ &\prod_{X \in \text{up } \mathcal{X}}^{\mathfrak{B}} (\mathcal{Y} \cap^{\mathfrak{B}} \langle f \rangle X) \neq \perp \Leftrightarrow \text{(corollary 545*)} \\ &\forall X \in \text{up } \mathcal{X} : \mathcal{Y} \cap^{\mathfrak{B}} \langle f \rangle X \neq \perp \Leftrightarrow \text{(theorem 515)} \\ &\forall X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y} : Y \cap^{\mathfrak{B}} \langle f \rangle X \neq \perp \Leftrightarrow \text{(corollary 514)} \\ &\forall X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y} : Y \cap^{\mathfrak{Z}_1} \langle f \rangle X \neq \perp \Leftrightarrow \\ &\forall X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y} : X [f] Y. \end{aligned}$$

* To apply corollary 545 we need to show that $\left\{ \frac{\mathcal{Y} \cap^{\mathfrak{B}} \langle f \rangle X}{X \in \text{up } \mathcal{X}} \right\}$ is a generalized filter base. To show it is enough to show that $\left\{ \frac{\langle f \rangle X}{X \in \text{up } \mathcal{X}} \right\}$ is a generalized filter base. But this easily follows from proposition 1391 and 551.

Similarly $\mathcal{X} \neq \langle \uparrow f^{-1} \rangle \mathcal{Y} \Leftrightarrow \forall X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y} : X [f] Y$. Thus $\mathcal{Y} \neq \langle \uparrow f \rangle \mathcal{X} \Leftrightarrow \mathcal{X} \neq \langle \uparrow f^{-1} \rangle \mathcal{Y}$. \square

PROPOSITION 1925. The above defined \uparrow is an injection.

PROOF. $\langle \uparrow f \rangle X = \prod_{X' \in \text{up } X}^{\mathfrak{B}} \langle f \rangle X' = \min_{X' \in \text{up } X} \langle f \rangle X' = \langle f \rangle X$. So $\langle f \rangle$ is determined by $\langle \uparrow f \rangle$. Likewise $\langle f^{-1} \rangle$ is determined by $\langle \uparrow f^{-1} \rangle$. \square

CONJECTURE 1926. (Non generalizing of theorem 1500) Pointfree funcoids f between some: a. atomistic but non-complete; b. complete but non-atomistic boolean lattices \mathfrak{Z}_0 and \mathfrak{Z}_1 do not bijectively correspond to morphisms $p \in \mathbf{Rel}(\text{atoms } \mathfrak{Z}_0, \text{atoms } \mathfrak{Z}_1)$ by the formulas:

$$\begin{aligned} \langle f \rangle X &= \bigsqcup \langle p \rangle^* \text{atoms } X, \quad \langle f^{-1} \rangle Y = \bigsqcup \langle p^{-1} \rangle^* \text{atoms } Y; \\ (x, y) \in \text{GR } p &\Leftrightarrow y \in \text{atoms } \langle f \rangle x \Leftrightarrow x \in \text{atoms } \langle f^{-1} \rangle y. \end{aligned}$$

Systems of sides

Now we will consider a common generalization of (some of pointfree) functors and (some of) Galois connections. The main purpose of this is general theorem 1974 below.

First consider some properties of Galois connections:

1. More on Galois connections

Here I will denote $\langle f \rangle$ the lower adjoint of a Galois connection f . **FiXme:** Switch to this notation in the book?

Let \mathbf{GAL} be the category of Galois connections. **FiXme:** Need to decide whether use $\mathbf{GAL}(A, B)$ or $A \otimes B$.

I will denote $(f, g)^{-1} = (g, f)$ for a Galois connection (f, g) .

We will order Galois connections by the formula

$$f \sqsubseteq g \Leftrightarrow \langle f \rangle \sqsubseteq \langle g \rangle \Leftrightarrow \langle f^{-1} \rangle \supseteq \langle g^{-1} \rangle.$$

OBVIOUS 1927. This defines a partial order on the set of Galois connections between any two (fixed) posets.

PROPOSITION 1928. If f and g are Galois connections (between a join-semilattice \mathfrak{A} and a meet-semilattice \mathfrak{B}), then there exists a Galois connection $f \sqcup g$ determined by the formula $\langle f \sqcup g \rangle x = \langle f \rangle x \sqcup \langle g \rangle x$.

PROOF. It is enough to prove that

$$(x \mapsto \langle f \rangle x \sqcup \langle g \rangle x, y \mapsto \langle f^{-1} \rangle y \sqcap \langle g^{-1} \rangle y)$$

is a Galois connection that is that

$$\langle f \rangle x \sqcup \langle g \rangle x \sqsubseteq y \Leftrightarrow x \sqsubseteq \langle f^{-1} \rangle y \sqcap \langle g^{-1} \rangle y$$

for all relevant x and y .

Really,

$$\begin{aligned} \langle f \rangle x \sqcup \langle g \rangle x \sqsubseteq y &\Leftrightarrow \langle f \rangle x \sqsubseteq y \wedge \langle g \rangle x \sqsubseteq y \Leftrightarrow \\ &x \sqsubseteq \langle f^{-1} \rangle y \wedge x \sqsubseteq \langle g^{-1} \rangle y \Leftrightarrow x \sqsubseteq \langle f^{-1} \rangle y \sqcap \langle g^{-1} \rangle y. \end{aligned}$$

□

FiXme: Describe infinite join of Galois connections.

PROPOSITION 1929. If \mathfrak{A} is a poset with least element, then $\langle a \rangle \perp = \perp$.

PROOF. $\langle a \rangle \perp \sqsubseteq y \Leftrightarrow \perp \sqsubseteq \langle a^{-1} \rangle y \Leftrightarrow 1$. Thus $\langle a \rangle \perp$ is the least element. □

PROPOSITION 1930. $(\mathfrak{A} \times \{\perp^{\mathfrak{B}}\}, \mathfrak{B} \times \{\top^{\mathfrak{A}}\})$ is the least Galois connection from a poset \mathfrak{A} with greatest element to a poset \mathfrak{B} with least element.

PROOF. Let's prove that it is a Galois connection. We need to prove

$$(\mathfrak{A} \times \{\perp^{\mathfrak{B}}\})x \sqsubseteq y \Leftrightarrow x \sqsubseteq (\mathfrak{B} \times \{\top^{\mathfrak{A}}\})y.$$

But this is trivially equivalent to $1 \Leftrightarrow 1$. Thus it's a Galois connection.

That it the least is obvious. □

COROLLARY 1931. $\langle \perp \rangle x = \perp$ for Galois connections from a poset \mathfrak{A} with greatest element to a poset \mathfrak{B} with least element. **FixMe: Clarify.**

THEOREM 1932. If \mathfrak{A} and \mathfrak{B} are bounded posets, then $\text{GAL}(\mathfrak{A}, \mathfrak{B})$ is bounded.

PROOF. That $\text{GAL}(\mathfrak{A}, \mathfrak{B})$ has least element was proved above. I will demonstrate that (α, β) is the greatest element of $\text{pFCD}(\mathfrak{A}, \mathfrak{B})$ for

$$\alpha X = \begin{cases} \perp^{\mathfrak{B}} & \text{if } X = \perp^{\mathfrak{A}} \\ \top^{\mathfrak{B}} & \text{if } X \neq \perp^{\mathfrak{A}} \end{cases}; \quad \beta Y = \begin{cases} \top^{\mathfrak{A}} & \text{if } Y = \top^{\mathfrak{B}} \\ \perp^{\mathfrak{A}} & \text{if } Y \neq \top^{\mathfrak{B}} \end{cases}.$$

First prove $Y \sqsubseteq \alpha X \Leftrightarrow X \sqsubseteq \beta Y$.

Really $\alpha X \sqsubseteq Y \Leftrightarrow X = \perp^{\mathfrak{A}} \vee Y = \top^{\mathfrak{B}} \Leftrightarrow X \sqsubseteq \beta Y$.

That it is the greatest Galois connection between \mathfrak{A} and \mathfrak{B} easily follows from proposition 1929. \square

THEOREM 1933. For every brouwerian lattice $x \mapsto c \sqcap x$ is a lower adjoint.

PROOF. By dual of theorem 148. \square

EXERCISE 1934. Describe the corresponding upper adjoint, especially for the special case of boolean lattices.

2. Definition

DEFINITION 1935. *System of presides* is a functor $\Upsilon = (f \mapsto \langle f \rangle)$ from an ordered category to the category of functions between (small) bounded lattices, such that (for all relevant variables):

- 1°. Every Hom-set of $\text{Src } \Upsilon$ is a bounded join-semilattice.
- 2°. $\langle a \rangle \perp = \perp$.
- 3°. $\langle a \sqcup b \rangle X = \langle a \rangle X \sqcup \langle b \rangle X$ (equivalent to Υ to be a join-semilattice homomorphism, if we order functions between small bounded lattices component-wise).

I call morphisms of such categories *sides*.¹

REMARK 1936. We could generalize to functions between small join-semilattices with least elements instead of bounded lattices only, but this is not really necessary.

DEFINITION 1937. I will call objects of the source category of this functor simply *objects of the presides*.

DEFINITION 1938. *Bounded system of presides* is system of presides from an ordered category with bounded Hom-sets such that $X, Y \in \text{Ob Src } \Upsilon$ the following additional axioms hold for all suitable a :

- 1°. $\langle \perp^{\text{Hom}(X, Y)} \rangle a = \perp$.
- 2°. $\langle \top^{\text{Hom}(X, Y)} \rangle a = \top$ unless $a = \perp$

DEFINITION 1939. *System of presides with identities* is a system of presides with a morphism $\text{id}_a \in \text{Src } \Upsilon$ for every object \mathfrak{A} of $\text{Src } \Upsilon$ and $a \in \mathfrak{A}$ and the following additional axioms:

- 1°. $\text{id}_c \sqsubseteq 1_{\mathfrak{A}}$ for every $c \in \mathfrak{A}$ where \mathfrak{A} is an object of $\text{Src } \Upsilon$.
- 2°. $\langle \text{id}_c \rangle = (\lambda x \in \mathfrak{A} : x \sqcap c)$ for every $c \in \mathfrak{A}$ where \mathfrak{A} is an object of $\text{Src } \Upsilon$

DEFINITION 1940. *System of sides* is a system of presides which is both bounded and with identities.

¹The idea for the name is that we consider one “side” $\langle f \rangle$ of a funcooid instead of both sides $\langle f \rangle$ and $\langle f^{-1} \rangle$.

PROPOSITION 1941. $\langle 1_{\mathfrak{A}}^{\text{Src } \Upsilon} \rangle a = a$ for every system of presides.

PROOF. By properties of functors. \square

DEFINITION 1942. I call a system of *monotone* presides a system of presides with additional axiom:

1°. $\langle a \rangle$ is monotone.

DEFINITION 1943. I call a system of *distributive* presides a system of presides with additional axiom:

1°. $\langle a \rangle (X \sqcup Y) = \langle a \rangle X \sqcup \langle a \rangle Y$.

OBVIOUS 1944. Every distributive system of presides is monotone.

PROPOSITION 1945. $\langle a \sqcap b \rangle X \sqsubseteq \langle a \rangle X \sqcap \langle b \rangle X$ for monotone systems of sides if Hom-sets are lattices.

DEFINITION 1946. A system of presides *with correct identities* is a system of presides with identities with additional axiom:

1°. $\text{id}_b \circ \text{id}_a = \text{id}_{a \sqcap b}$.

PROPOSITION 1947. Every faithful system of presides with identities is with correct identities.

PROOF. $\langle \text{id}_b \circ \text{id}_a \rangle x = (\langle \text{id}_b \rangle \circ \langle \text{id}_a \rangle)x = \langle \text{id}_b \rangle \langle \text{id}_a \rangle x = b \sqcap a \sqcap x = \langle \text{id}_{b \sqcap a} \rangle x$. Thus by faithfulness $\text{id}_b \circ \text{id}_a = \text{id}_{b \sqcap a} = \text{id}_{a \sqcap b}$. \square

DEFINITION 1948. *Restricting* a side f to an object X is defined by the formula $f|_X = f \circ \text{id}_X$.

DEFINITION 1949. *Image* of a preside is defined by the formula $\text{im } f = \langle f \rangle \top$.

DEFINITION 1950. Protofunctors *over* a set X of functors is a protofunctor f such that $\langle f \rangle \in X \wedge \langle f^{-1} \rangle \in X$.

3. Concrete examples of sides

OBVIOUS 1951. The category \mathbf{Rel} with $\langle f \rangle = \langle f \rangle^*$ for $f \in \mathbf{Rel}$ and usual id_c defines a distributive system of sides with correct identities.

3.1. Some subsides.

DEFINITION 1952. *Full subsystem* of a system Υ of presides is the functor Υ restricted to a full subcategory of $\text{Src } \Upsilon$.

OBVIOUS 1953. Full subsystem of a system of presides is always a system of presides.

OBVIOUS 1954. Full subsystem of a bounded system of presides is always a bounded subsystem of presides.

OBVIOUS 1955.

1°. Full subsystem of a system of presides with identities is always with identities.

2°. Full subsystem of a system of presides with correct identities is always with correct identities.

OBVIOUS 1956. Full subsystem of a distributive system of presides is always a distributive system of presides.

OBVIOUS 1957. Full subsystem of a system of sides is always a system of sides.

3.2. Functors and pointfree functors.

PROPOSITION 1958. The category of pointfree functors between starrish join-semilattices with usual $\langle f \rangle$ defines a system of presides.

PROOF. Theorem 1420. □

PROPOSITION 1959. The category of pointfree functors between bounded starrish join-semilattices with usual $\langle f \rangle$ defines a system of bounded presides.

PROOF. Take the proof of theorem 1417 into account. □

PROPOSITION 1960. The category of pointfree functors from a starrish join-semilattices to a separable starrish join-semilattices defines a distributive system of presides.

PROOF. Theorem 1392. □

PROPOSITION 1961. The category of pointfree functors between starrish lattices with usual $\langle f \rangle$ and usual id_c defines a system of presides with correct identities.

PROOF. That it is with identities is obvious.

That it is with correct identities is obvious. □

OBVIOUS 1962. The category of pointfree functors between bounded starrish lattices with usual $\langle f \rangle$ and usual id_c defines a system of sides with correct identities.

PROPOSITION 1963. The category of functors with usual $\langle f \rangle$ and usual id_c defines a system of sides with correct identities.

PROOF. Because it can be considered a full subsystem of the category of point-free functors between bounded starrish lattices with usual $\langle f \rangle$. □

3.3. Galois connections.

PROPOSITION 1964. The category of Galois connections between (small) lattices with least elements together with usual $\langle f \rangle$ defines a distributive system of presides.

PROOF. Propositions 1928 and 1929 for a system of presides.

It is distributive because lower adjoints preserve all joins. □

PROPOSITION 1965. The category of Galois connections between (small) bounded lattices together with usual $\langle f \rangle$ defines a bounded system of presides.

PROOF. Theorem 1932. □

PROPOSITION 1966. The category of Galois connections between (small) Heyting lattices together with usual $\langle f \rangle$ defines a system of sides with correct identities.

PROOF. Theorem 1933 ensures that they a system of sides with identities. The identities are correct due to faithfulness. □

3.4. Reloids.

PROPOSITION 1967. Reloids with the functor $f \mapsto \langle (\text{FCD})f \rangle$ and usual id_c form a system of sides with correct identities.

PROOF. It is really a functor because $\langle (\text{FCD})g \rangle \circ \langle (\text{FCD})f \rangle = \langle (\text{FCD})g \circ (\text{FCD})f \rangle = \langle (\text{FCD})(g \circ f) \rangle$ for every composable reloids f and g .

$$\langle a \rangle \perp = \langle (\text{FCD})a \rangle \perp = \perp;$$

$$\begin{aligned} \langle a \sqcup b \rangle X &= \langle (\text{FCD})(a \sqcup b) \rangle X = \langle (\text{FCD})a \sqcup (\text{FCD})b \rangle X = \\ & \langle (\text{FCD})a \rangle X \sqcup \langle (\text{FCD})b \rangle X = \langle a \rangle X \sqcup \langle b \rangle X; \end{aligned}$$

thus it is a system of presides.

That this is a bounded system of presides follows from the formulas $(\text{FCD})_{\perp}^{\text{RLD}(A,B)} = \perp$ and $(\text{FCD})_{\top}^{\text{RLD}(A,B)} = \top$.

It is with identities, because proposition 982. It is with correct identities by proposition 942. \square

FiXme: Also for pointfree reلودs.

FiXme: These examples works for (dagger) systems of sides with binary product.

4. Product

DEFINITION 1968. *Binary product* of objects of presides with identities is defined by the formula $X \times Y = \text{id}_Y \circ \top \circ \text{id}_X$.

DEFINITION 1969. System of presides with identities is *with correct binary product* when $f \sqcap (X \times Y) = \text{id}_Y \circ f \circ \text{id}_X$ for every preside f .

PROPOSITION 1970. $\langle A \times B \rangle X = \begin{cases} \perp & \text{if } X \simeq A \\ B & \text{if } X \not\simeq A \end{cases}$

PROOF.

$$\begin{aligned} \langle A \times B \rangle X &= \langle \text{id}_B \circ \top \circ \text{id}_A \rangle X = \langle \text{id}_B \rangle \langle \top \rangle \langle \text{id}_A \rangle X = \\ &= B \sqcap \langle \top \rangle (X \sqcap A) = B \sqcap \begin{cases} \perp & \text{if } X \simeq A \\ \top & \text{if } X \not\simeq A \end{cases} = \begin{cases} \perp & \text{if } X \simeq A \\ B & \text{if } X \not\simeq A \end{cases} \end{aligned}$$

\square

DEFINITION 1971. I will call a system of sides *with correct meet* when

$$(X_0 \times Y_0) \sqcap (X_1 \times Y_1) = (X_0 \sqcap X_1) \times (Y_0 \sqcap Y_1).$$

PROPOSITION 1972. Faithful systems of presides with identities are with correct meet.

PROOF. $(X_0 \times Y_0) \sqcap (X_1 \times Y_1) = \text{id}_{Y_1} \circ (X_0 \times Y_0) \circ \text{id}_{X_1}$. Thus

$$\begin{aligned} \langle (X_0 \times Y_0) \sqcap (X_1 \times Y_1) \rangle P &= \langle \text{id}_{Y_1} \rangle \langle X_0 \times Y_0 \rangle \langle \text{id}_{X_1} \rangle P = \\ &= \langle \text{id}_{Y_1} \rangle \begin{cases} \perp & \text{if } X_0 \simeq \langle \text{id}_{X_1} \rangle P \\ Y_0 & \text{if } X_0 \not\simeq \langle \text{id}_{X_1} \rangle P \end{cases} = \begin{cases} \perp & \text{if } X_0 \sqcap X_1 \simeq P \\ Y_0 \sqcap Y_1 & \text{if } X_0 \sqcap X_1 \not\simeq P \end{cases} = \\ &= \langle (X_0 \sqcap X_1) \times (Y_0 \sqcap Y_1) \rangle P. \end{aligned}$$

So $(X_0 \times Y_0) \sqcap (X_1 \times Y_1) = (X_0 \sqcap X_1) \times (Y_0 \sqcap Y_1)$ follows by full faithfulness. \square

PROPOSITION 1973. Systems of presides with correct identities are with correct meet.

PROOF. $(X_0 \times Y_0) \sqcap (X_1 \times Y_1) = \text{id}_{Y_1} \circ (X_0 \times Y_0) \circ \text{id}_{X_1} = \text{id}_{Y_1} \circ (\text{id}_{Y_0} \circ \top \circ \text{id}_{X_0}) \circ \text{id}_{X_1} = \text{id}_{Y_0 \sqcap Y_1} \circ \top \circ \text{id}_{X_0 \sqcap X_1} = (X_0 \sqcap X_1) \times (Y_0 \sqcap Y_1)$. \square

For some sides holds the formula $f \circ (X \times Y) = X \times \langle f \rangle Y$. I refrain to give a name for this property.

5. Negative results

The following negative result generalizes theorem 3.8 in [3].

THEOREM 1974. The element $1^{(\text{Src } \Upsilon)(\mathfrak{A}, \mathfrak{A})}$ is not complemented if \mathfrak{A} is a non-atomic boolean lattice, for every monotone system of sides.

PROOF. Let $T = 1^{(\text{Src } \Upsilon)(\mathfrak{A}, \mathfrak{A})}$.

Let's suppose $T \sqcup V = \top$ for $V \in (\text{Src } \Upsilon)(\mathfrak{A}, \mathfrak{A})$ and prove $T \sqcap V \neq \perp$.

Then $\langle T \sqcup V \rangle a = \top$ for all $a \neq \perp$ and thus $\langle V \rangle a \sqcup a = \top$.

Consequently $\langle V \rangle a \sqsupseteq \neg a$ for all $a \neq \perp$.

If a isn't an atom, then there exists b with $0 \sqsubset b \sqsubset a$ and hence $\langle V \rangle a \sqsupseteq \langle V \rangle b \sqsupseteq \neg b \sqsupseteq \neg a$; thus $\langle V \rangle a \sqsupseteq \neg a$.

There is such $c \sqsubset \top$ that $a \sqsubseteq c$ for every atom a . (Really, suppose some element $p \neq \perp$ has no atoms. Thus all atoms are in $\neg p$.)

For $a \not\sqsubseteq c$ we have $\langle V \rangle a \sqcap a \sqsubset \perp$ for all $a \sqsubseteq \neg c$ thus $\langle T \sqcap V \rangle a \sqsupseteq \langle V \rangle a \sqcap a \sqsubset \perp$.

Thus $\langle (T \sqcap V) \circ \text{id}_{\neg c} \rangle a \sqsubset \perp$

So $T \sqcap V \sqsupseteq (T \sqcap V) \circ \text{id}_{\neg c} \sqsubset \perp$. So V is not a complement of T . \square

COROLLARY 1975. $(\text{Src } \Upsilon)(\mathfrak{A}, \mathfrak{A})$ is not boolean if \mathfrak{A} is a non-atomic boolean lattice.

6. Dagger systems of sides

PROPOSITION 1976.

- 1°. For a partially ordered dagger category, each of Hom-set of which has least element, we have $\perp^\dagger = \perp$.
- 2°. For a partially ordered dagger category, each of Hom-set of which has greatest element, we have $\top^\dagger = \top$.

PROOF. $\forall f \in \text{Hom}(A, B) : \perp^\dagger \sqsubseteq f \Leftrightarrow \forall f \in \text{Hom}(A, B) : \perp \sqsubseteq f^\dagger \Leftrightarrow \forall f \in \text{Hom}(A, B) : \perp \sqsubseteq f \Leftrightarrow 1$. Thus \perp^\dagger is the least.

The other items is dual. \square

DEFINITION 1977. *Dagger system of presides with identities* is system of pre-sides with identities with category $\text{Src } \Upsilon$ being a partially ordered dagger category and $(\text{id}_X)^\dagger = \text{id}_X$ for every X .

PROPOSITION 1978. For a system of sides we have $(X \times Y)^\dagger = Y \times X$.

PROOF. $(X \times Y)^\dagger = (\text{id}_Y \circ \top \circ \text{id}_X)^\dagger = \text{id}_X^\dagger \circ \top^\dagger \circ \text{id}_Y^\dagger = \text{id}_X \circ \top \circ \text{id}_Y = Y \times X$. \square

FiXme: Which properties of pointfree funcoids can be generalized for sides?

Backward Functors

This is a preliminary partial draft.

Fix a family \mathfrak{A} of posets.

DEFINITION 1979. Let f be a staroid of filters $\mathfrak{F}(\mathfrak{A}_i)$ on boolean lattices \mathfrak{A}_i . *Backward functor* for the argument $k \in \text{dom } \mathfrak{A}$ of f is the functor $\text{Back}(f, k)$ defined by the formula (for every $X \in \mathfrak{A}_k$)

$$\langle \text{Back}(f, k) \rangle X = \left\{ \frac{L \in \prod_{i \in \text{dom } \mathfrak{A}} \mathfrak{F}(\mathfrak{A}_i)}{X \in \langle f \rangle_k L} \right\}.$$

PROPOSITION 1980. Backward functor is properly defined.

PROOF. $\langle \text{Back}(f, k) \rangle^*(X \sqcup Y) = \left\{ \frac{L \in \prod \mathfrak{A}}{X \sqcup Y \in \langle f \rangle_k L} \right\} = \left\{ \frac{L \in \prod \mathfrak{A}}{X \in \langle f \rangle_k L \vee Y \in \langle f \rangle_k L} \right\} = \left\{ \frac{L \in \prod \mathfrak{A}}{X \in \langle f \rangle_k L} \right\} \cup \left\{ \frac{L \in \prod \mathfrak{A}}{Y \in \langle f \rangle_k L} \right\} = \langle \text{Back}(f, k) \rangle^* X \cup \langle \text{Back}(f, k) \rangle^* Y. \quad \square$

OBVIOUS 1981. Backward functor is co-complete.

PROPOSITION 1982. If f is a principal staroid then $\text{Back}(f, k)$ is a complete functor.

PROOF. ?? □

PROPOSITION 1983. f can be restored from $\text{Back}(f, k)$ (for every fixed k).

PROOF. ?? □

PROPOSITION 1984. $f \mapsto \text{Back}(f, k)$ is an order isomorphism $\text{Strd}^{\mathfrak{A}} \rightarrow \text{FCD}(\mathfrak{A}_k, \text{Strd}^{(\text{dom } \mathfrak{A}) \setminus \{k\}})$.

PROOF. ?? □

Cauchy Filters on Reloids

In this chapter I consider *low filters* on reloids, generalizing Cauchy filters on uniform spaces. Using low filters, I define Cauchy-complete reloids, generalizing complete uniform spaces.

FiXme: I forgot to note that Cauchy spaces induce topological (or convergence) spaces.

1. Preface

Replace `\langle ... \rangle` with `\supfun{...}` in L^AT_EX.

This is a preliminary partial draft.

To understand this article you need first look into my book [2].

<http://math.stackexchange.com/questions/401989/>

[what-are-interesting-properties-of-totally-bounded-uniform-spaces](http://math.stackexchange.com/questions/401989/what-are-interesting-properties-of-totally-bounded-uniform-spaces)

http://ncatlab.org/nlab/show/proximity+space#uniform_spaces for a proof sketch that proximities correspond to totally bounded uniformities.

2. Low spaces

FiXme: Analyze <http://link.springer.com/article/10.1007/s10474-011-0136-9> (“A note on Cauchy spaces”), <http://link.springer.com/article/10.1007/BF00873992> (“Filter spaces”). It also contains references to some useful results, including (“On continuity structures and spaces of mappings” freely available at <https://eudml.org/doc/16128>) that the category FIL of filter spaces is isomorphic to the category of filter merotopic spaces (copy its definition).

DEFINITION 1985. A *lower set*¹ of filters on U (a set) is a set \mathcal{C} of filters on U , such that if $\mathcal{G} \sqsubseteq \mathcal{F}$ and $\mathcal{F} \in \mathcal{C}$ then $\mathcal{G} \in \mathcal{C}$.

REMARK 1986. Note that we are particularly interested in nonempty (= containing the improper filter) lower sets of filters. This does not match the traditional theory of Cauchy spaces (see below) which are traditionally defined as not containing empty set. Allowing them to contain empty set has some advantages:

- Meet of any lower filters is a lower filter.
- Some formulas become a little simpler.

DEFINITION 1987. I call *low space* a set together with a nonempty lower set of filters on this set. Elements of a (given) low space are called *Cauchy filters*.

DEFINITION 1988. $\text{GR}(U, \mathcal{C}) = \mathcal{C}$; $\text{Ob}(U, \mathcal{C}) = U$. $\text{GR}(U, \mathcal{C})$ is read as *graph of space* (U, \mathcal{C}) . I denote $\text{Low}(U)$ the set of graphs of low spaces on the set U . Similarly I will denote its subsets $\text{ASJ}(U)$, $\text{CASJ}(U)$, $\text{Cau}(U)$, $\text{CCau}(U)$ (see below).

FiXme: Should use “space structure” instead of “graph of space”, to match customary terminology.

¹Remember that our orders on filters is the reverse to set theoretic inclusion. It could be called an *upper set* in other sources.

DEFINITION 1989. Introduce an order on graphs of low spaces and on low spaces: $\mathcal{C} \sqsubseteq \mathcal{D} \Leftrightarrow \mathcal{C} \subseteq \mathcal{D}$ and $(U, \mathcal{C}) \sqsubseteq (U, \mathcal{D}) \Leftrightarrow \mathcal{C} \subseteq \mathcal{D}$.

OBVIOUS 1990. Every set of low spaces on some set is partially ordered.

3. Almost sub-join-semilattices

DEFINITION 1991. For a join-semilattice \mathfrak{A} , a *almost sub-join-semilattice* is such a set $S \in \mathcal{P}\mathfrak{A}$, that if $\mathcal{F}, \mathcal{G} \in S$ and $\mathcal{F} \not\sqsubseteq \mathcal{G}$ then $\mathcal{F} \sqcup \mathcal{G} \in S$.

DEFINITION 1992. For a complete lattice \mathfrak{A} , a *completely almost sub-join-semilattice* is such a set $S \in \mathcal{P}\mathfrak{A}$, that if $\prod T \neq \perp^{\mathcal{F}(X)}$ then $\prod T \in S$ for every $T \in \mathcal{P}S$.

OBVIOUS 1993. Every completely almost sub-join-semilattice is a almost sub-join-semilattice.

4. Cauchy spaces

DEFINITION 1994. A *reflexive* low space is a low space (U, \mathcal{C}) such that $\forall x \in U : \uparrow^U \{x\} \in \mathcal{C}$.

DEFINITION 1995. The *identity* low space $1^{\text{Low}(U)}$ on a set U is the low space with graph $\left\{ \frac{\uparrow^U \{x\}}{x \in U} \right\}$.

OBVIOUS 1996. A low space f is reflexive iff $f \supseteq 1^{\text{Low}(\text{Ob } f)}$.

DEFINITION 1997. An *almost sub-join space* is a low space whose graph is an almost sub-join-semilattice. I will denote such spaces as **ASJ**.

DEFINITION 1998. A *completely almost sub-join space* is a low space whose graph is a completely almost sub-join-semilattice. I will denote such spaces as **CASJ**.

DEFINITION 1999. A *precauchy space* (aka *filter space*) is a reflexive low space. I will denote such spaces as **preCau**.

DEFINITION 2000. A *Cauchy space* is a precauchy space which is also an almost sub-join space. I will denote such spaces as **Cau**.

DEFINITION 2001. A *completely Cauchy space* is a precauchy space which is also a completely almost sub-join space. I will denote such spaces as **CCau**.

OBVIOUS 2002. Every completely Cauchy space is a Cauchy space.

PROPOSITION 2003. $a \sqcup \left\{ \frac{\mathcal{X} \in \mathcal{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\} b = a \sqcup b$ for $a, b \in \left\{ \frac{\mathcal{X} \in \mathcal{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\}$, provided that \mathcal{F} is a proper fixed Cauchy filter on an almost sub-join space.

PROOF. \mathcal{F} is proper. So we have $a \sqcap b \sqsupseteq \mathcal{F} \neq \perp^{\mathcal{F}(X)}$. Thus $a \sqcup b$ is a Cauchy filter and so $a \sqcup b \in \left\{ \frac{\mathcal{X} \in \mathcal{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\}$. \square

PROPOSITION 2004. $\prod \left\{ \frac{\mathcal{X} \in \mathcal{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\} S = \prod S$ for nonempty $S \in \mathcal{P} \left\{ \frac{\mathcal{X} \in \mathcal{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\}$, provided that \mathcal{F} is a proper fixed Cauchy filter on a completely almost sub-join space.

PROOF. \mathcal{F} is proper. So for every nonempty $S \in \mathcal{P} \left\{ \frac{\mathcal{X} \in \mathcal{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\}$ we have $\prod S \sqsupseteq \mathcal{F} \neq \perp^{\mathcal{F}(X)}$. Thus $\prod S$ is a Cauchy filter and so $\prod S \in \left\{ \frac{\mathcal{X} \in \mathcal{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\}$. \square

COROLLARY 2005. Every proper Cauchy filter is contained in a unique maximal Cauchy filter (for completely almost sub-join spaces).

PROOF. Let \mathcal{F} be a proper Cauchy filter. Then $\bigsqcup \left\{ \frac{\mathcal{X} \in \mathcal{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\} \left\{ \frac{\mathcal{X} \in \mathcal{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\}$ (existing by the above proposition) is the maximal Cauchy filter containing \mathcal{F} .

Suppose another maximal Cauchy filter \mathcal{T} contains \mathcal{F} . Then $\mathcal{T} \in \left\{ \frac{\mathcal{X} \in \mathcal{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\}$ and thus $\mathcal{T} = \bigsqcup \left\{ \frac{\mathcal{X} \in \mathcal{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\} \left\{ \frac{\mathcal{X} \in \mathcal{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\}$. \square

5. Relationships with symmetric reloids

FiXme: Also consider relationships with funcoids.

DEFINITION 2006. Denote $(\text{RLD})_{\text{Low}}(U, \mathcal{C}) = \bigsqcup \left\{ \frac{\mathcal{X} \times^{\text{RLD}} \mathcal{X}}{\mathcal{X} \in \mathcal{C}} \right\}$.

DEFINITION 2007. $(\text{Low})\nu$ (*low space* for endoreloid ν) is a low space on U such that

$$\text{GR}(\text{Low})\nu = \left\{ \frac{\mathcal{X} \in \mathcal{F}(U)}{\mathcal{X} \times^{\text{RLD}} \mathcal{X} \sqsubseteq \nu} \right\}.$$

THEOREM 2008. If (U, \mathcal{C}) is a low space, then $(U, \mathcal{C}) = (\text{Low})(\text{RLD})_{\text{Low}}(U, \mathcal{C})$.

PROOF. If $\mathcal{X} \in \mathcal{C}$ then $\mathcal{X} \times^{\text{RLD}} \mathcal{X} \sqsubseteq (\text{RLD})_{\text{Low}}(U, \mathcal{C})$ and thus $\mathcal{X} \in \text{GR}(\text{Low})(\text{RLD})_{\text{Low}}(U, \mathcal{C})$. Thus $(U, \mathcal{C}) \sqsubseteq (\text{Low})(\text{RLD})_{\text{Low}}(U, \mathcal{C})$.

Let's prove $(U, \mathcal{C}) \sqsupseteq (\text{Low})(\text{RLD})_{\text{Low}}(U, \mathcal{C})$.

Let $\mathcal{A} \in \text{GR}(\text{Low})(\text{RLD})_{\text{Low}}(U, \mathcal{C})$. We need to prove $\mathcal{A} \in \mathcal{C}$.

Really $\mathcal{A} \times^{\text{RLD}} \mathcal{A} \sqsubseteq (\text{RLD})_{\text{Low}}(U, \mathcal{C})$. It is enough to prove that $\exists \mathcal{X} \in \mathcal{C} : \mathcal{A} \sqsubseteq \mathcal{X}$.

Suppose $\nexists \mathcal{X} \in \mathcal{C} : \mathcal{A} \sqsubseteq \mathcal{X}$.

For every $\mathcal{X} \in \mathcal{C}$ obtain $X_{\mathcal{X}} \in \mathcal{X}$ such that $X_{\mathcal{X}} \notin \mathcal{A}$ (if for all $X \in \mathcal{X}$ we have $X_{\mathcal{X}} \in \mathcal{A}$, then $\mathcal{X} \sqsupseteq \mathcal{A}$ what is contrary to our supposition).

It is now enough to prove $\mathcal{A} \times^{\text{RLD}} \mathcal{A} \not\sqsubseteq \bigsqcup \left\{ \frac{\uparrow^U X_{\mathcal{X}} \times^{\text{RLD}} \uparrow^U X_{\mathcal{X}}}{\mathcal{X} \in \mathcal{C}} \right\}$.

Really, $\bigsqcup \left\{ \frac{\uparrow^U X_{\mathcal{X}} \times^{\text{RLD}} \uparrow^U X_{\mathcal{X}}}{\mathcal{X} \in \mathcal{C}} \right\} = \uparrow^{\text{RLD}(U, U)} \bigcup \left\{ \frac{\uparrow^U X_{\mathcal{X}} \times^{\text{RLD}} \uparrow^U X_{\mathcal{X}}}{\mathcal{X} \in \mathcal{C}} \right\}$. So our claim takes the form $\bigcup \left\{ \frac{\uparrow^U X_{\mathcal{X}} \times^{\text{RLD}} \uparrow^U X_{\mathcal{X}}}{\mathcal{X} \in \mathcal{C}} \right\} \not\sqsubseteq \text{GR}(\mathcal{A} \times^{\text{RLD}} \mathcal{A})$ that is $\forall A \in \mathcal{A} : \bigcup \left\{ \frac{\uparrow^U X_{\mathcal{X}} \times^{\text{RLD}} \uparrow^U X_{\mathcal{X}}}{\mathcal{X} \in \mathcal{C}} \right\} \not\sqsupseteq A \times A$ what is true because $X_{\mathcal{X}} \not\sqsupseteq A$ for every $A \in \mathcal{A}$. \square

REMARK 2009. The last theorem does not hold with $\mathcal{X} \times^{\text{FCD}} \mathcal{X}$ instead of $\mathcal{X} \times^{\text{RLD}} \mathcal{X}$ (take $\mathcal{C} = \left\{ \frac{\{x\}}{x \in U} \right\}$ for an infinite set U as a counter-example).

REMARK 2010. Not every symmetric reloid is in the form $(\text{RLD})_{\text{Low}}(U, \mathcal{C})$ for some Cauchy space (U, \mathcal{C}) . The same Cauchy space can be induced by different uniform spaces. See <http://math.stackexchange.com/questions/702182/different-uniform-spaces-having-the-same-set-of-cauchy-filters>

PROPOSITION 2011.

1°. $(\text{Low})f$ is reflexive iff endoreloid f is reflexive.

2°. $(\text{RLD})_{\text{Low}}f$ is reflexive iff low space f is reflexive.

PROOF.

1°. f is reflexive $\Leftrightarrow 1^{\text{RLD}} \sqsubseteq f \Leftrightarrow \forall x \in \text{Ob } f : \uparrow(\{x\} \times \{x\}) \sqsubseteq f \Leftrightarrow \forall x \in \text{Ob } f : \uparrow\{x\} \times^{\text{RLD}} \uparrow\{x\} \sqsubseteq f \Leftrightarrow \forall x \in \text{Ob } f : \uparrow\{x\} \in (\text{Low})f \Leftrightarrow (\text{Low})f$ is reflexive.

2°. Let f is reflexive. Then $\forall x \in \text{Ob } f : \uparrow\{x\} \in f; \forall x \in \text{Ob } f : \uparrow\{x\} \times^{\text{RLD}} \uparrow\{x\} \sqsubseteq (\text{RLD})_{\text{Low}}f; \forall x \in \text{Ob } f : \uparrow(\{x\} \times \{x\}) \sqsubseteq (\text{RLD})_{\text{Low}}f; 1^{\text{RLD}} \sqsubseteq (\text{RLD})_{\text{Low}}f$.

Let now $(\text{RLD})_{\text{Low}}f$ be reflexive. Then $f = (\text{Low})(\text{RLD})_{\text{Low}}f$ is reflexive. \square

DEFINITION 2012. A *transitive* low space is such low space f that $(\text{RLD})_{\text{Low}} f \circ (\text{RLD})_{\text{Low}} f = (\text{RLD})_{\text{Low}} f$.

REMARK 2013. The composition $(\text{RLD})_{\text{Low}} f \circ (\text{RLD})_{\text{Low}} f$ may be inequal to $(\text{RLD})_{\text{Low}} \mu$ for all low spaces μ (exercise!). Thus usefulness of the concept of transitive low spaces is questionable.

REMARK 2014. Every low space is “symmetric” in the sense that it corresponds to a symmetric reloid.

THEOREM 2015. (Low) is the upper adjoint of $(\text{RLD})_{\text{Low}}$.

PROOF. We will prove $(\text{Low})(\text{RLD})_{\text{Low}} f \sqsupseteq f$ and $(\text{RLD})_{\text{Low}}(\text{Low})g \sqsubseteq g$ (that (Low) and $(\text{RLD})_{\text{Low}}$ are monotone is obvious).

Really:

$$\begin{aligned} \text{GR}(\text{Low})(\text{RLD})_{\text{Low}} f &= \text{GR}(\text{Low}) \bigsqcup \left\{ \frac{\mathcal{X} \times^{\text{RLD}} \mathcal{X}}{\mathcal{X} \in \text{GR } f} \right\} = \\ &= \left\{ \frac{\mathcal{Y} \in \mathcal{F} \text{ Ob}(f)}{\mathcal{Y} \times^{\text{RLD}} \mathcal{Y} \sqsubseteq \bigsqcup \left\{ \frac{\mathcal{X} \times^{\text{RLD}} \mathcal{X}}{\mathcal{X} \in \text{GR } f} \right\}} \right\} \supseteq \left\{ \frac{\mathcal{Y} \in \mathcal{F} \text{ Ob}(f)}{\mathcal{Y} \in \text{GR } f} \right\} = \text{GR } f; \\ (\text{RLD})_{\text{Low}}(\text{Low})g &= \bigsqcup \left\{ \frac{\mathcal{X} \times^{\text{RLD}} \mathcal{X}}{\mathcal{X} \in \text{GR}(\text{Low})g} \right\} = \bigsqcup \left\{ \frac{\mathcal{X} \times^{\text{RLD}} \mathcal{X}}{\mathcal{X} \in \mathcal{F}(\text{Ob } g), \mathcal{X} \times^{\text{RLD}} \mathcal{X} \sqsubseteq g} \right\} \sqsubseteq g. \quad \square \end{aligned}$$

COROLLARY 2016.

- 1°. $(\text{RLD})_{\text{Low}} \bigsqcup S = \bigsqcup \langle (\text{RLD})_{\text{Low}} \rangle^* S$;
- 2°. $(\text{Low}) \bigsqcap S = \bigsqcap \langle (\text{Low}) \rangle^* S$.

Below it's proved that (Low) and $(\text{RLD})_{\text{Low}}$ can be restricted to completely almost sub-join spaces and symmetrically transitive reloids. Thus they preserve joins of (completely) almost sub-join spaces and meets of symmetrically transitive reloids. **FixMe: Check. FixMe: Move it to be below the definition.**

6. Lattices of low spaces

PROPOSITION 2017. $\mu \sqsubseteq \nu \Leftrightarrow \forall \mathcal{X} \in \text{GR } \mu \exists \mathcal{Y} \in \text{GR } \nu : \mathcal{X} \sqsubseteq \mathcal{Y}$ for low filter spaces (on the same set U).

PROOF.

- $$\begin{aligned} \Rightarrow. \mu \sqsubseteq \nu &\Leftrightarrow \text{GR } \mu \subseteq \text{GR } \nu \Rightarrow \forall \mathcal{X} \in \text{GR } \mu \exists \mathcal{Y} \in \text{GR } \nu : \mathcal{X} = \mathcal{Y} \Rightarrow \forall \mathcal{X} \in \text{GR } \mu \exists \mathcal{Y} \in \\ &\text{GR } \nu : \mathcal{X} \sqsubseteq \mathcal{Y}. \\ \Leftarrow. \text{Let } \forall \mathcal{X} \in \text{GR } \mu \exists \mathcal{Y} \in \text{GR } \nu : \mathcal{X} \sqsubseteq \mathcal{Y}. &\text{ Take } \mathcal{X} \in \text{GR } \mu. \text{ Then } \exists \mathcal{Y} \in \text{GR } \nu : \mathcal{X} \sqsubseteq \\ &\mathcal{Y}. \text{ Thus } \mathcal{X} \in \text{GR } \nu. \text{ So } \text{GR } \mu \subseteq \text{GR } \nu \text{ that is } \mu \sqsubseteq \nu. \end{aligned}$$

□

OBVIOUS 2018.

- 1°. $(\text{RLD})_{\text{Low}}$ is an order embedding.
- 2°. (Low) is an order homomorphism.

I will denote $\bigsqcup, \bigsqcap, \sqcup, \sqcap$ the lattice operations on low spaces or graphs of low spaces.

PROPOSITION 2019. $\bigsqcup S = \bigcup S$ for every set S of graphs of low spaces on some set.

PROOF. It's enough to prove that there is a low space μ such that $\text{GR } \mu = \bigcup S$. In other words, it's enough to prove that $\bigcup S$ is a nonempty lower set, but that's obvious. **FixMe: A little more detailed proof.** □

PROPOSITION 2020. $\prod S = \left\{ \frac{\prod \text{im } P}{P \in \prod_{X \in S} X} \right\}$ for every set S of graphs of low spaces on some set.

PROOF. First prove that there is such low space μ that $\mu = \left\{ \frac{\prod \text{im } P}{P \in \prod_{X \in S} X} \right\}$. In other words, we need to prove that $\left\{ \frac{\prod \text{im } P}{P \in \prod_{X \in S} X} \right\}$ is a nonempty lower set. That it is nonempty is obvious. Let filter $\mathcal{G} \sqsubseteq \mathcal{F}$ and $\mathcal{F} \in \left\{ \frac{\prod \text{im } P}{P \in \prod_{X \in S} X} \right\}$. Then $\mathcal{F} = \prod \text{im } P$ for a $P \in \prod_{X \in S} X$ that is $P(X) \in X$ for every $X \in S$. Take $P' = (\mathcal{G} \sqcap) \circ P$. Then $P' \in \prod_{X \in S} X$ because $P'(X) \in X$ for every $X \in S$ and thus obviously $\mathcal{G} = \prod \text{im } P'$ and thus $\mathcal{G} \in \left\{ \frac{\prod \text{im } P}{P \in \prod_{X \in S} X} \right\}$. So such μ exists.

It remains to prove that μ is the greatest lower bound of S .

μ is a lower bound of S . Really, let $X \in S$ and $Y \in X$. Then exists $P \in \prod_{X \in S} X$ such that $P(X) = Y$ (taken into account that every X is nonempty) and thus $\text{im } P \ni Y$ and so $\prod \text{im } P \sqsubseteq Y$, that is (proposition 2017) $\mu \sqsubseteq X$.

Let ν be a lower bound of S . It remains to prove that $\mu \sqsupseteq \nu$, that is $\forall Q \in \nu : Q = \prod \text{im } P$ for some $P \in \prod_{X \in S} X$. Take $P = (\lambda X \in S : Q)$. This $P \in \prod_{X \in S} X$ because $Q \in X$ for every $X \in S$. \square

COROLLARY 2021. $f \sqcap g = \left\{ \frac{F \sqcap G}{F \in f, G \in g} \right\}$ for every graphs f and g of low spaces (on some set).

6.1. Its subsets.

PROPOSITION 2022. The set of sub-join low spaces (on some fixed set) is meet-closed in the lattice of low spaces on a set.

PROOF. Let f, g be graphs of almost sub-join spaces (on some fixed set), $f \sqcap g = \left\{ \frac{F \sqcap G}{F \in f, G \in g} \right\}$.

If $\mathcal{A}, \mathcal{B} \in f \sqcap g$ and $\mathcal{A} \not\sqsubseteq \mathcal{B}$, then $\mathcal{A}, \mathcal{B} \in f$ and $\mathcal{A}, \mathcal{B} \in g$. Thus $\mathcal{A} \sqcup \mathcal{B} \in f$ and $\mathcal{A} \sqcup \mathcal{B} \in g$ and so $\mathcal{A} \sqcup \mathcal{B} \in f \sqcap g$. \square

COROLLARY 2023. The set of Cauchy spaces (on some fixed set), is meet-closed in the lattice of low spaces on a set.

PROPOSITION 2024. The set of completely almost sub-join spaces is meet-closed in the lattice of low spaces on a set.

PROOF. Let S be a set of graphs of almost completely sub-join low spaces (on some fixed set). $\prod S = \left\{ \frac{\prod \text{im } P}{P \in \prod_{X \in S} X} \right\}$.

If $\mathcal{A}, \mathcal{B} \in \prod S$ and $\mathcal{A} \not\sqsubseteq \mathcal{B}$, then $\mathcal{A}, \mathcal{B} \in X$ for every $X \in S$. Thus $\mathcal{A} \sqcup \mathcal{B} \in X$ and so $\mathcal{A} \sqcup \mathcal{B} \in \prod S$. \square

COROLLARY 2025. The set of completely Cauchy spaces is meet-closed in the lattice of low spaces on a set.

From the above it follows:

OBVIOUS 2026. The following sets are complete lattices in our order:

- 1°. almost sub-join spaces, whose graphs are almost sub-join-semilattices;
- 2°. completely almost sub-join spaces;
- 3°. reflexive low spaces;
- 4°. precauchy spaces;

- 5°. Cauchy spaces;
6°. completely Cauchy spaces.

Denote $Z(f) = \left\{ \frac{F \sqcup G}{F \in f, G \in f, F \not\leq G} \right\} \cup \{\perp\}$ for every set f of filters (on some fixed set).

PROPOSITION 2027. $Z(f) \sqsupseteq f$ for every set f of filters.

PROOF. Consider for $F \in f$ both cases $F = \perp$ and $F \neq \perp$. □

LEMMA 2028. For graphs of low spaces f, g (on the same set)

$$Q = \bigcup S \cup Z\left(\bigcup S\right) \cup Z\left(Z\left(\bigcup S\right)\right) \cup \dots$$

is a graph of some almost sub-join space.

PROOF. That it is nonempty and a lower set of filters is obvious. It remains to prove that it is an almost sub-join-semilattice.

Let $\mathcal{A}, \mathcal{B} \in Q$ and $\mathcal{A} \not\leq \mathcal{B}$. Then

$$\mathcal{A}, \mathcal{B} \in \underbrace{Z \dots Z}_{n \text{ times}}\left(\bigcup S\right)$$

for a natural n . Thus

$$\mathcal{A} \sqcup \mathcal{B} \in \underbrace{Z \dots Z}_{n+1 \text{ times}}\left(\bigcup S\right)$$

and so $\mathcal{A} \sqcup \mathcal{B} \in Q$. □

PROPOSITION 2029. Join on the lattice of graphs of almost sub-join spaces is described by the formula

$$\bigsqcup^{\text{ASJ}} S = \bigcup S \cup Z\left(\bigcup S\right) \cup Z\left(Z\left(\bigcup S\right)\right) \cup \dots$$

PROOF. The right part of the above formula μ is a graph of an almost sub-join space (lemma).

That μ is an upper bound of S is obvious.

It remains to prove that μ is the least upper bound.

Suppose ν is an upper bound of S . Then $\nu \supseteq \bigcup S$. Thus, because ν is an almost sub-join-semilattice, $Z(\nu) \subseteq \nu$, likewise $Z(Z(\nu)) \subseteq \nu$, etc. Consequently $Z(\bigcup S) \subseteq \nu$, $Z(Z(\bigcup S)) \subseteq \nu$, etc. So we have $\mu \sqsubseteq \nu$. □

PROPOSITION 2030. **FiXme: Should be merged with the previous proposition.**

$$\bigsqcup^{\text{ASJ}} S = \left\{ \frac{F_0 \sqcup \dots \sqcup F_{n-1}}{F_0, \dots, F_{n-1} \in \bigcup S, F_0 \not\leq F_1 \wedge F_1 \not\leq F_2 \wedge \dots \wedge F_{n-2} \not\leq F_{n-1} \text{ for } n \in \mathbb{N}} \right\}.$$

REMARK 2031. We take $F_0 \sqcup \dots \sqcup F_{n-1} = \perp$ for $n = 0$.

PROOF. Denote the right part of the above formula as R .

Suppose $F \in R$. Let's prove by induction that $F \in Q$. If $F = \perp$ that's obvious. Suppose we know that $F_0 \sqcup \dots \sqcup F_{n-1} \in Q$ that is for a natural m

$$F_0 \sqcup \dots \sqcup F_{n-1} \in \underbrace{Z \dots Z}_{m \text{ times}}\left(\bigcup S\right)$$

for $F_0, \dots, F_{n-1} \in \bigcup S$, $F_0 \not\leq F_1 \wedge F_1 \not\leq F_2 \wedge \dots \wedge F_{n-2} \not\leq F_{n-1}$ and also $F_{n-1} \not\leq F_n$. Then $F_0 \sqcup \dots \sqcup F_{n-1} \not\leq F_n$ and thus $F_0 \sqcup \dots \sqcup F_{n-1} \sqcup F_n \in \underbrace{Z \dots Z}_{m+1 \text{ times}}\left(\bigcup S\right)$ that is

$F_0 \sqcup \dots \sqcup F_{n-1} \sqcup F_n \in Q$. So $F \in Q$ for every $F \in R$.

Now suppose $F \in Q$ that is for a natural m

$$F \in \underbrace{Z \dots Z}_{m \text{ times}} \left(\bigcup S \right).$$

Let's prove by induction that $F = F_0 \sqcup \dots \sqcup F_{n-1}$ for some $F_0, \dots, F_{n-1} \in \bigcup S$ such that $F_0 \not\prec F_1 \wedge F_1 \not\prec F_2 \wedge \dots \wedge F_{n-2} \not\prec F_{n-1}$. If $m = 0$ then $F \in \bigcup S$ and our promise is obvious. Let our statement holds for a natural m . Prove that it holds for

$$F' \in \underbrace{Z \dots Z}_{m+1 \text{ times}} \left(\bigcup S \right).$$

We have $F' = Z(F)$ for some $F = F_0 \sqcup \dots \sqcup F_{n-1}$ where $F_0 \not\prec F_1 \wedge F_1 \not\prec F_2 \wedge \dots \wedge F_{n-2} \not\prec F_{n-1}$. The case $F' = \perp$ is easy. So we can assume $F' = A \sqcup B$ where $A, B \in F$ and $A \not\prec B$. By the statement of induction $A = A_0 \sqcup \dots \sqcup A_{p-1}$, $B = B_0 \sqcup \dots \sqcup B_{q-1}$ for natural p and q , where $A_0 \not\prec A_1 \wedge A_1 \not\prec A_2 \wedge \dots \wedge A_{p-2} \not\prec A_{p-1}$, $B_0 \not\prec B_1 \wedge B_1 \not\prec B_2 \wedge \dots \wedge B_{q-2} \not\prec B_{q-1}$. Take j such that $A \not\prec B_j$ and then take i such that $A_i \not\prec B_j$. Then (using symmetry of the relation $\not\prec$) we have $(A_0 \not\prec A_1 \wedge A_1 \not\prec A_2 \wedge \dots \wedge A_{p-2} \not\prec A_{p-1}) \wedge (A_{p-1} \not\prec A_{p-2} \not\prec \dots \wedge A_{i+1} \not\prec A_i) \wedge A_i \not\prec B_j \wedge (B_j \not\prec B_{j-1} \wedge \dots \wedge B_1 \not\prec B_0) \wedge (B_0 \not\prec B_1 \wedge B_1 \not\prec B_2 \wedge \dots \wedge B_{q-2} \not\prec B_{q-1})$. So $F' = A \sqcup B$ is representable as the join of a finite sequence of filters with each adjacent pair of filters in this sequence being intersecting. That is $F' \in Q$. \square

PROPOSITION 2032. The lattice of Cauchy spaces (on some set) is a complete sublattice of the lattice of almost sub-join spaces.

PROOF. It's obvious, taking into account obvious 1996. \square

$$\text{Denote } Z_\infty(f) = \left\{ \frac{\bigsqcup T}{T \in \mathcal{P}f \wedge \prod T \neq \perp} \right\} \cup \{\perp\}.$$

PROPOSITION 2033. $Z_\infty(f) \supseteq f$.

PROOF. Consider for $F \in f$ both cases $F = \perp$ and $F \neq \perp$. \square

LEMMA 2034. If S is a set of graphs of low spaces, then

$$Q = \bigcup S \cup Z_\infty \left(\bigcup S \right) \cup Z_\infty \left(Z_\infty \left(\bigcup S \right) \right) \cup \dots$$

is a graph of a completely Cauchy space.

PROOF. That it is nonempty and a lower set of filters is obvious. It remains to prove that it is a completely almost sub-join-semilattice.

Let $T \in \mathcal{P}Q$ and $\prod T \neq \perp$. Then

$$T \in \mathcal{P} \underbrace{Z_\infty \dots Z_\infty}_{n \text{ times}} \left(\bigcup S \right)$$

for a natural n . Thus

$$T \in \mathcal{P} \underbrace{Z_\infty \dots Z_\infty}_{n+1 \text{ times}} \left(\bigcup S \right)$$

and so $\bigsqcup T \in Q$. \square

PROPOSITION 2035. The lattice of completely Cauchy spaces (on some set) is a complete sublattice of the lattice of completely almost sub-join spaces.

PROOF. It's obvious, taking into account obvious 1996. \square

PROPOSITION 2036. Join of a set S on the lattice of graphs of completely almost sub-join-semilattice is described by the formula:

$$\text{CASJ} \bigsqcup S = \bigcup S \cup Z_\infty \left(\bigcup S \right) \cup Z_\infty \left(Z_\infty \left(\bigcup S \right) \right) \cup \dots$$

PROOF. The right part of the above formula μ is a graph of an almost sub-join space (lemma).

That μ is an upper bound of S is obvious.

It remains to prove that μ is the least upper bound.

Suppose ν is an upper bound of S . Then $\nu \supseteq \bigcup S$. Thus, because ν is an almost sub-join-semilattice, $Z_\infty(\nu) \subseteq \nu$, likewise $Z_\infty(Z_\infty(\nu)) \subseteq \nu$, etc. Consequently $Z_\infty(\bigcup S) \subseteq \nu$, $Z_\infty(Z_\infty(\bigcup S)) \subseteq \nu$, etc. So we have $\mu \sqsubseteq \nu$. \square

CONJECTURE 2037.

$$1^\circ. \bigsqcup^{\text{CASJ}} S = \left\{ \frac{\bigsqcup T_0 \sqcup \dots \sqcup T_{n-1}}{n \in \mathbb{N}, T_0, \dots, T_{n-1} \in \bigcup S,} \right\};$$

$$\left. \begin{array}{l} \prod T_0 \neq \perp \wedge \dots \wedge \prod T_{n-1} \neq \perp, \\ \bigsqcup T_0 \not\leq \bigsqcup T_1 \wedge \dots \wedge \bigsqcup T_{n-2} \not\leq \bigsqcup T_{n-1}. \end{array} \right\}$$

$$2^\circ. \bigsqcup^{\text{CASJ}} S = \left\{ \frac{\bigsqcup T_0 \sqcup \bigsqcup T_1 \sqcup \dots}{T_0, T_1, \dots \in \bigcup S,} \right\}$$

$$\left. \begin{array}{l} \prod T_0 \neq \perp \wedge \prod T_2 \neq \perp \wedge \dots, \bigsqcup T_0 \not\leq \bigsqcup T_1 \wedge \bigsqcup T_1 \not\leq \bigsqcup T_2 \wedge \dots \end{array} \right\}.$$

7. Up-complete low spaces

DEFINITION 2038. *Ideal base* is a nonempty subset S of a poset such that $\forall a, b \in S \exists c \in S : (a, b \sqsubseteq c)$.

OBVIOUS 2039. Ideal base is dual of filter base.

THEOREM 2040. Product of nonempty posets is a ideal base iff every factor is an ideal base.

PROOF. [FiXme: more detailed proof](#)

In one direction it is easy: Suppose one multiplier is not a dcpo. Take a chain with fixed elements (thanks our posets are nonempty) from other multipliers and for this multiplier take the values which form a chain without the join. This proves that the product is not a dcpo.

Let now every factor is dcpo. S is a filter base in $\prod \mathfrak{A}$ iff each component is a filter base. Each component has a join. Thus by proposition 611 S has a componentwise join. \square

DEFINITION 2041. I call a low space *up-complete* when each ideal base (or equivalently every nonempty chain, see theorem 560) in this space has join in this space.

REMARK 2042. Elements of this ideal base are filters. (Thus is could be called a generalized ideal base.)

EXAMPLE 2043.

1 $^\circ$. $\left\{ \frac{\mathcal{X} \in \mathfrak{F}[0; +\infty[}{\exists \varepsilon > 0: \mathcal{X} \sqsubseteq \uparrow \varepsilon; +\infty[} \right\} \cup \uparrow \{0\}$ is a graph of Cauchy space on \mathbb{R}_+ , but not up-complete.

2 $^\circ$. $\mathfrak{F}[0; +\infty[$ is a strictly greater graph of Cauchy space on \mathbb{R}_+ and is up-complete.

LEMMA 2044. Let f be a reloid. Each ideal base $T \subseteq \left\{ \frac{(A,B)}{\mathcal{A} \times^{\text{RLD}} \mathcal{B} \sqsubseteq f} \right\}$ has a join in this set.

PROOF. Let T be an ideal base and $\forall (A, B) \in T : \mathcal{A} \times^{\text{FCD}} \mathcal{B} \sqsubseteq f$.

$\forall (A, B) \in T \forall \mathcal{X} \in \mathcal{F} \text{ Src } f : (\mathcal{X} \not\prec \mathcal{A} \Rightarrow \mathcal{B} \sqsubseteq \langle f \rangle \mathcal{X})$;

taking join we have:

$\forall \mathcal{A} \in \text{Pr}_0 T \forall \mathcal{X} \in \mathcal{F} \text{ Src } f : (\mathcal{X} \not\prec \mathcal{A} \Rightarrow \bigsqcup_{\mathcal{B} \in \text{Pr}_1 T} \mathcal{B} \sqsubseteq \langle f \rangle \mathcal{X})$;

$\forall \mathcal{A} \in \text{Pr}_0 T : \mathcal{A} \times^{\text{FCD}} \bigsqcup_{\mathcal{B} \in \text{Pr}_1 T} \mathcal{B} \sqsubseteq f$.

Now repeat a similar operation second time:

$\forall \mathcal{A} \in \text{Pr}_0 T : \bigsqcup_{\mathcal{B} \in \text{Pr}_1 T} \mathcal{B} \times^{\text{FCD}} \mathcal{A} \sqsubseteq f^{-1}$;

$\forall \mathcal{A} \in \text{Pr}_0 T \forall \mathcal{Y} \in \mathcal{F} \text{ Dst } f : (\mathcal{Y} \not\prec \bigsqcup_{\mathcal{B} \in \text{Pr}_1 T} \mathcal{B} \Rightarrow \mathcal{A} \sqsubseteq \langle f^{-1} \rangle \mathcal{Y})$;

$\forall \mathcal{Y} \in \mathcal{F} \text{ Dst } f : (\mathcal{Y} \not\prec \bigsqcup_{\mathcal{B} \in \text{Pr}_1 T} \mathcal{B} \Rightarrow \bigsqcup_{\mathcal{A} \in \text{Pr}_0 T} \mathcal{A} \sqsubseteq \langle f^{-1} \rangle \mathcal{Y})$;

$\bigsqcup_{\mathcal{B} \in \text{Pr}_1 T} \mathcal{B} \times^{\text{FCD}} \bigsqcup_{\mathcal{A} \in \text{Pr}_0 T} \mathcal{A} \sqsubseteq f^{-1}$;

$\bigsqcup_{\mathcal{A} \in \text{Pr}_0 T} \mathcal{A} \times^{\text{FCD}} \bigsqcup_{\mathcal{B} \in \text{Pr}_1 T} \mathcal{B} \sqsubseteq f$. But $\bigsqcup_{\mathcal{A} \in \text{Pr}_0 T} \mathcal{A} \times^{\text{FCD}} \bigsqcup_{\mathcal{B} \in \text{Pr}_1 T} \mathcal{B}$ is the join in consideration, because ideal base is ideal base in each argument. \square

PROPOSITION 2045. A Cauchy space generated by an endoreloid is always up-complete.

PROOF. Let f be an endoreloid. $\text{GR}(\text{Low})f = \left\{ \frac{\mathcal{X} \in \text{Ob } f}{\mathcal{X} \times^{\text{RLD}} \mathcal{X} \sqsubseteq f} \right\}$.

Let $T \subseteq \left\{ \frac{\mathcal{X} \in \text{Ob } f}{\mathcal{X} \times^{\text{RLD}} \mathcal{X} \sqsubseteq f} \right\}$ be an ideal base.

Then $N = \left\{ \frac{(\mathcal{F}, \mathcal{F})}{\mathcal{F} \in T} \right\}$ is also an ideal base. Obviously $N \subseteq \left\{ \frac{(A, B)}{\mathcal{A} \times^{\text{RLD}} \mathcal{B} \sqsubseteq f} \right\}$. Thus by the lemma it has a join in $\left\{ \frac{(A, B)}{\mathcal{A} \times^{\text{RLD}} \mathcal{B} \sqsubseteq f} \right\}$. It's easy to see that this join is in $\left\{ \frac{(A, A)}{\mathcal{A} \in \text{Ob } f, \mathcal{A} \times^{\text{RLD}} \mathcal{A} \sqsubseteq f} \right\}$. Consequently T has a join in $\left\{ \frac{\mathcal{X} \in \text{Ob } f}{\mathcal{X} \times^{\text{RLD}} \mathcal{X} \sqsubseteq f} \right\}$. \square

It is long time known that (using our terminology) low space induced by a uniform space is a Cauchy space, but that it is complete and up-complete is probably first discovered by Victor Porton.

8. More on Cauchy filters

OBVIOUS 2046. Low filter on an endoreloid ν is a filter \mathcal{F} such that

$$\forall U \in \text{GR } f \exists A \in \mathcal{F} : A \times A \subseteq U.$$

REMARK 2047. The above formula is the standard definition of Cauchy filters on uniform spaces.

PROPOSITION 2048. If $\nu \sqsupseteq \nu \circ \nu^{-1}$ then every neighborhood filter is a Cauchy filter, that it

$$\nu \sqsupseteq \langle (\text{FCD})\nu \rangle^* \{x\} \times^{\text{RLD}} \langle (\text{FCD})\nu \rangle^* \{x\}$$

for every point x .

PROOF. $\langle (\text{FCD})\nu \rangle^* \{x\} \times^{\text{RLD}} \langle (\text{FCD})\nu \rangle^* \{x\} = \langle (\text{FCD})\nu \rangle \uparrow^{\text{Ob } \nu} \{x\} \times^{\text{RLD}} \langle (\text{FCD})\nu \rangle \uparrow^{\text{Ob } \nu} \{x\} = \nu \circ (\uparrow^{\text{Ob } \nu} \{x\} \times^{\text{RLD}} \uparrow^{\text{Ob } \nu} \{x\}) \circ \nu^{-1} = \nu \circ (\uparrow^{\text{RLD}(\text{Ob } \nu, \text{Ob } \nu)} \{x, x\}) \circ \nu^{-1} \sqsubseteq \nu \circ \text{id}^{\text{RLD}(\text{Ob } \nu, \text{Ob } \nu)} \circ \nu^{-1} = \nu \circ \nu^{-1} \sqsubseteq \nu$. \square

PROPOSITION 2049. If $\nu \sqsupseteq \nu \circ \nu^{-1}$ a filter converges (in ν) to a point, it is a low filter, provided that every neighborhood filter is a low filter.

PROOF. Let $\mathcal{F} \sqsubseteq \langle (\text{FCD})\nu \rangle^* \{x\}$. Then $\mathcal{F} \times^{\text{RLD}} \mathcal{F} \sqsubseteq \langle (\text{FCD})\nu \rangle^* \{x\} \times^{\text{RLD}} \langle (\text{FCD})\nu \rangle^* \{x\} \sqsubseteq \nu$. \square

COROLLARY 2050. If a filter converges to a point, it is a low filter, provided that $\nu \sqsupseteq \nu \circ \nu^{-1}$.

9. Maximal Cauchy filters

LEMMA 2051. Let S be a set of sets with $\prod \langle \uparrow^{\mathfrak{F}} \rangle^* S \neq 0^{\mathfrak{F}}$ (in other words, S has finite intersection property). Let $T = \left\{ \frac{X \times X}{X \in S} \right\}$. Then

$$\bigcup T \circ \bigcup T = \bigcup S \times \bigcup S.$$

PROOF. Let $x \in \bigcup S$. Then $x \in X$ for some $X \in S$. $\langle \bigcup T \rangle \{x\} \supseteq \uparrow X \supseteq \bigcap S \neq \emptyset$. Thus

$$\langle \bigcup T \circ \bigcup T \rangle \{x\} = \langle \bigcup T \rangle \langle \bigcup T \rangle \{x\} \in \langle \uparrow^{\text{FCD}} \bigcup T \rangle \prod \langle \uparrow^{\mathfrak{F}} \rangle S \supseteq \bigsqcup \left\{ \frac{\langle \uparrow^{\text{FCD}}(X \times X) \rangle \prod \langle \uparrow^{\mathfrak{F}} \rangle S}{X \in S} \right\} = \bigsqcup \left\{ \frac{\uparrow^{\mathfrak{F}} X}{X \in S} \right\} = \bigsqcup \langle \uparrow^{\mathfrak{F}} \rangle S \text{ that is } \langle \bigcup T \circ \bigcup T \rangle \{x\} \supseteq \bigcup S. \quad \square$$

COROLLARY 2052. Let S be a set of filters (on some fixed set) with nonempty meet. Let

$$T = \left\{ \frac{\mathcal{X} \times^{\text{RLD}} \mathcal{X}}{\mathcal{X} \in S} \right\}$$

Then

$$\bigsqcup T \circ \bigsqcup T = \bigsqcup S \times^{\text{RLD}} \bigsqcup S.$$

$$\text{PROOF. } \bigsqcup T \circ \bigsqcup T = \prod \left\{ \frac{\uparrow^{\mathfrak{F}}(X \circ X)}{X \in \bigsqcup T} \right\}.$$

If $X \in \bigsqcup T$ then $X = \bigcup_{Q \in T} (P_Q \times P_Q)$ where $P_Q \in Q$. Therefore by the lemma we have

$$\bigcup \left\{ \frac{P_Q \times P_Q}{Q \in T} \right\} \circ \bigcup \left\{ \frac{P_Q \times P_Q}{Q \in T} \right\} = \bigcup_{Q \in T} P_Q \times \bigcup_{Q \in T} P_Q.$$

Thus $X \circ X = \bigcup_{Q \in T} P_Q \times \bigcup_{Q \in T} P_Q$.

$$\text{Consequently } \bigsqcup T \circ \bigsqcup T = \prod \left\{ \frac{\uparrow^{\mathfrak{F}}(\bigcup_{Q \in T} P_Q \times \bigcup_{Q \in T} P_Q)}{X \in \bigsqcup T} \right\} \supseteq \bigsqcup S \times^{\text{RLD}} \bigsqcup S.$$

$$\bigsqcup T \circ \bigsqcup T \sqsubseteq \bigsqcup S \times^{\text{RLD}} \bigsqcup S \text{ is obvious.} \quad \square$$

DEFINITION 2053. I call an endoreloid ν *symmetrically transitive* iff for every symmetric endofunctor $f \in \text{FCD}(\text{Ob } \nu, \text{Ob } \nu)$ we have $f \sqsubseteq \nu \Rightarrow f \circ f \sqsubseteq \nu$.

OBVIOUS 2054. It is symmetrically transitive if at least one of the following holds:

- 1°. $\nu \circ \nu \sqsubseteq \nu$;
- 2°. $\nu \circ \nu^{-1} \sqsubseteq \nu$;
- 3°. $\nu^{-1} \circ \nu \sqsubseteq \nu$.
- 4°. $\nu^{-1} \circ \nu^{-1} \sqsubseteq \nu$.

COROLLARY 2055. Every uniform space is symmetrically transitive.

PROPOSITION 2056. $(\text{Low})\nu$ is a completely Cauchy space for every symmetrically transitive endoreloid ν .

$$\text{PROOF. Suppose } S \in \mathcal{P} \left\{ \frac{\mathcal{X} \in \mathfrak{F}}{\mathcal{X} \times^{\text{RLD}} \mathcal{X} \sqsubseteq \nu} \right\}.$$

$\bigsqcup \left\{ \frac{\mathcal{X} \times^{\text{RLD}} \mathcal{X}}{\mathcal{X} \in S} \right\} \sqsubseteq \nu$; $\bigsqcup \left\{ \frac{\mathcal{X} \times^{\text{RLD}} \mathcal{X}}{\mathcal{X} \in S} \right\} \circ \bigsqcup \left\{ \frac{\mathcal{X} \times^{\text{RLD}} \mathcal{X}}{\mathcal{X} \in S} \right\} \sqsubseteq \nu$; $\bigsqcup S \times^{\text{RLD}} \bigsqcup S \sqsubseteq \nu$ (taken into account that S has nonempty meet). Thus $\bigsqcup S$ is Cauchy. \square

PROPOSITION 2057. The neighbourhood filter $\langle (\text{FCD})\nu \rangle^* \{x\}$ of a point $x \in \text{Ob } \nu$ is a maximal Cauchy filter, if it is a Cauchy filter and ν is a reflexive reloid.

FiXme: Does it holds for all low filters?

PROOF. Let $\mathcal{N} = \langle (\text{FCD})\nu \rangle^* \{x\}$. Let $\mathcal{C} \sqsupseteq \mathcal{N}$ be a Cauchy filter. We need to show $\mathcal{N} \sqsupseteq \mathcal{C}$.

Since \mathcal{C} is Cauchy filter, $\mathcal{C} \times^{\text{RLD}} \mathcal{C} \sqsubseteq \nu$. Since $\mathcal{C} \sqsupseteq \mathcal{N}$ we have \mathcal{C} is a neighborhood of x and thus $\uparrow^{\text{Ob}\nu} \{x\} \sqsubseteq \mathcal{C}$ (reflexivity of ν). Thus $\uparrow^{\text{Ob}\nu} \{x\} \times^{\text{RLD}} \mathcal{C} \sqsubseteq \mathcal{C} \times^{\text{RLD}} \mathcal{C}$ and hence $\uparrow^{\text{Ob}\nu} \{x\} \times^{\text{RLD}} \mathcal{C} \sqsubseteq \nu$;

$$\mathcal{C} \sqsubseteq \text{im}(\nu|_{\uparrow^{\text{Ob}\nu} \{x\}}) = \langle (\text{FCD})\nu \rangle^* \{x\} = \mathcal{N}.$$

□

10. Cauchy continuous functions

DEFINITION 2058. A function $f : U \rightarrow V$ is *Cauchy continuous* from a low space (U, \mathcal{C}) to a low space (V, \mathcal{D}) when $\forall \mathcal{X} \in \mathcal{C} : \langle \uparrow^{\text{FCD}} f \rangle \mathcal{X} \in \mathcal{D}$.

PROPOSITION 2059. Let f be a principal reloid. Then $f \in \text{C}((\text{RLD})_{\text{Low}}\mathcal{C}, (\text{RLD})_{\text{Low}}\mathcal{D})$ iff f is Cauchy continuous.

$$\begin{aligned} f \circ (\text{RLD})_{\text{Low}}\mathcal{C} \circ f^{-1} \sqsubseteq (\text{RLD})_{\text{Low}}\mathcal{D} &\Leftrightarrow \\ \bigsqcup_{\mathcal{X} \in \mathcal{C}} (f \circ (\mathcal{X} \times^{\text{RLD}} \mathcal{X}) \circ f^{-1}) \sqsubseteq (\text{RLD})_{\text{Low}}\mathcal{D} &\Leftrightarrow \\ \bigsqcup_{\mathcal{X} \in \mathcal{C}} (\langle \uparrow^{\text{FCD}} f \rangle \mathcal{X} \times^{\text{RLD}} \langle \uparrow^{\text{FCD}} f \rangle \mathcal{X}) \sqsubseteq (\text{RLD})_{\text{Low}}\mathcal{D} &\Leftrightarrow \\ \forall \mathcal{X} \in \mathcal{C} : \langle \uparrow^{\text{FCD}} f \rangle \mathcal{X} \times^{\text{RLD}} \langle \uparrow^{\text{FCD}} f \rangle \mathcal{X} \sqsubseteq (\text{RLD})_{\text{Low}}\mathcal{D} &\Leftrightarrow \\ \forall \mathcal{X} \in \mathcal{C} : \langle \uparrow^{\text{FCD}} f \rangle \mathcal{X} \in \mathcal{D}. & \end{aligned}$$

Thus we have expressed Cauchy properties through the algebra of reلودs.

11. Cauchy-complete reلودs

DEFINITION 2060. An endoreloid ν is *Cauchy-complete* iff every low filter for this reloid converges to a point.

REMARK 2061. In my book [2] *complete reloid* means something different. I will always prepend the word ‘‘Cauchy’’ to the word ‘‘complete’’ when meaning is by the last definition.

https://en.wikipedia.org/wiki/Complete_uniform_space#Completeness

12. Totally bounded

<http://ncatlab.org/nlab/show/Cauchy+space>

DEFINITION 2062. Low space is called *totally bounded* when every proper filter contains a proper Cauchy filter.

OBVIOUS 2063. A reloid ν is totally bounded iff

$$\forall X \in \mathcal{D} \text{ Ob}\nu \exists \mathcal{X} \in \mathfrak{F}^{\text{Ob}\nu} : (\perp \neq \mathcal{X} \sqsubseteq \uparrow^{\text{Ob}\nu} X \wedge \mathcal{X} \times^{\text{RLD}} \mathcal{X} \sqsubseteq \nu).$$

THEOREM 2064. A symmetric transitive reloid is totally bounded iff its Cauchy space is totally bounded.

PROOF.

\Rightarrow . Let \mathcal{F} be a proper filter on $\text{Ob}\nu$ and let $a \in \text{atoms}\mathcal{F}$. It’s enough to prove that a is Cauchy.

Let $D \in \text{GR}\nu$. Let also $E \in \text{GR}\nu$ is symmetric and $E \circ E \subseteq D$. There exists a finite subset $F \subseteq \text{Ob}\nu$ such that $\langle E \rangle F = \text{Ob}\nu$. Then obviously exists $x \in F$ such that $a \sqsubseteq \uparrow^{\text{Ob}\nu} \langle E \rangle \{x\}$, but $\langle E \rangle \{x\} \times \langle E \rangle \{x\} = E^{-1} \circ (\{x\} \times \{x\}) \circ E \subseteq D$, thus $a \times^{\text{RLD}} a \sqsubseteq \uparrow^{\text{RLD}(\text{Ob}\nu, \text{Ob}\nu)} D$.

Because D was taken arbitrary, we have $a \times^{\text{RLD}} a \sqsubseteq \nu$ that is a is Cauchy.

\Leftarrow . Suppose that Cauchy space associated with a reloid ν is totally bounded but the reloid ν isn't totally bounded. So there exists a $D \in \text{GR } \nu$ such that $(\text{Ob } \nu) \setminus \langle D \rangle F \neq \emptyset$ for every finite set F .

Consider the filter base

$$S = \left\{ \frac{(\text{Ob } \nu) \setminus \langle D \rangle F}{F \in \mathcal{P} \text{Ob } \nu, F \text{ is finite}} \right\}$$

and the filter $\mathcal{F} = \prod \langle \uparrow^{\text{Ob } \nu} \rangle S$ generated by this base. The filter \mathcal{F} is proper because intersection $P \cap Q \in S$ for every $P, Q \in S$ and $\emptyset \notin S$. Thus there exists a Cauchy (for our Cauchy space) filter $\mathcal{X} \sqsubseteq \mathcal{F}$ that is $\mathcal{X} \times^{\text{RLD}} \mathcal{X} \sqsubseteq \nu$.

Thus there exists $M \in \mathcal{X}$ such that $M \times M \subseteq D$. Let F be a finite subset of $\text{Ob } \nu$. Then $(\text{Ob } \nu) \setminus \langle D \rangle F \in \mathcal{F} \sqsupseteq \mathcal{X}$. Thus $M \not\subseteq (\text{Ob } \nu) \setminus \langle D \rangle F$ and so there exists a point $x \in M \cap ((\text{Ob } \nu) \setminus \langle D \rangle F)$.

$\langle M \times M \rangle \{p\} \subseteq \langle D \rangle \{x\}$ for every $p \in M$; thus $M \subseteq \langle D \rangle \{x\}$.

So $M \subseteq \langle D \rangle (F \cup \{x\})$. But this means that $M \in \mathcal{X}$ does not intersect $(\text{Ob } \nu) \setminus \langle D \rangle (F \cup \{x\}) \in \mathcal{F} \sqsupseteq \mathcal{X}$, what is a contradiction (taken into account that \mathcal{X} is proper). □

<http://math.stackexchange.com/questions/104696/pre-compactness-total-boundedness-and-cauchy-sequential-compactness>

13. Totally bounded funcoids

DEFINITION 2065. A funcoid ν is totally bounded iff

$$\forall X \in \text{Ob } \nu \exists \mathcal{X} \in \mathfrak{F}^{\text{Ob } \nu} : (0 \neq \mathcal{X} \sqsubseteq \uparrow^{\text{Ob } \nu} X \wedge \mathcal{X} \times^{\text{FCD}} \mathcal{X} \sqsubseteq \nu).$$

This can be rewritten in elementary terms (without using funcoidal product: $\mathcal{X} \times^{\text{FCD}} \mathcal{X} \sqsubseteq \nu \Leftrightarrow \forall P \in \partial \mathcal{X} : \mathcal{X} \sqsubseteq \langle \nu \rangle P \Leftrightarrow \forall P \in \partial \mathcal{X}, Q \in \partial \mathcal{X} : P [\nu]^* Q \Leftrightarrow \forall P, Q \in \text{Ob } \nu : (\forall E \in \mathcal{X} : (E \cap P \neq \emptyset \wedge E \cap Q \neq \emptyset) \Rightarrow P [\nu]^* Q)$.

Note that probably I am the first person which has written the above formula (for proximity spaces for instance) explicitly.

14. On principal low spaces

DEFINITION 2066. A low space (U, \mathcal{C}) is *principal* when all filters in \mathcal{C} are principal.

PROPOSITION 2067. Having fixed a set U , principal reflexive low spaces on U bijectively correspond to principal reflexive symmetric endoreloids on U .

PROOF. ??

<http://math.stackexchange.com/questions/701684/union-of-cartesian-squares> □

15. Rest

https://en.wikipedia.org/wiki/Cauchy_filter#Cauchy_filters

https://en.wikipedia.org/wiki/Uniform_space “Hausdorff completion of a uniform space” here)

<http://at.yorku.ca/z/a/a/b/13.htm> : the category **Prox** of proximity spaces and proximally continuous maps (i.e. maps preserving nearness between two sets) is isomorphic to the category of totally bounded uniform spaces (and uniformly continuous maps).

https://en.wikipedia.org/wiki/Cauchy_space <http://ncatlab.org/nlab/show/Cauchy+space>
<http://arxiv.org/abs/1309.1748>
http://projecteuclid.org/download/pdf_1/euclid.pja/1195521991
http://www.emis.de/journals/HOA/IJMMS/Volume5_3/404620.pdf
[~/math/books/Cauchy_spaces.pdf](#)
<https://ncatlab.org/nlab/show/Cauchy+space> defines compact Cauchy spaces!
<http://www.hindawi.com/journals/ijmms/1982/404620/abs/> (open access article) describes criteria for a Cauchy space to be uniformizable.

On “Each regular paratopological group is completely regular” article

In this chapter I attempt to rewrite the paper [1] in more general setting of functors and reloids. I attempt to construct a “royal road” to finding proofs of statements of this paper and similar ones, what is important because we lose 60 years waiting for any proof.

1. Definition of normality

By definition (slightly generalizing the special case if μ is a quasi-uniform space from [1]) a pair of an endo-reloid μ and a complete functor ν (playing role of a generalization of a topological space) on a set U is *normal* when

$$\langle \nu^{-1} \rangle^* A \sqsubseteq \langle \nu^{-1 \circ} \rangle^* \langle \nu^{-1} \rangle^* \langle F \rangle^* A$$

for every entourage $F \in \text{up } \mu$ of μ and every set $A \subseteq U$.

Note that this is *not* the same as customary definition of normal topological spaces.

THEOREM 2068. An endoreloid μ is normal on endoreloid ν iff

$$\nu \circ \nu^{-1} \sqsubseteq \nu^{-1} \circ (\text{FCD})\mu.$$

PROOF. Equivalently transforming the criterion of normality (which should hold for all $F \in \text{up } \mu$) using proposition 1922:

$$\langle \nu \rangle^* \langle \nu^{-1} \rangle^* A \sqsubseteq \langle \nu^{-1} \rangle^* \langle F \rangle^* A.$$

Also note

$$\prod_{F \in \text{up } \mu}^{\mathcal{F}} \langle \nu^{-1} \rangle^* \langle F \rangle^* A = (\text{because functors preserve filtered meets}) = \langle \nu^{-1} \rangle^* \prod_{F \in \text{up } \mu}^{\mathcal{F}} \langle F \rangle^* A = \langle \nu^{-1} \rangle^* \langle (\text{FCD})\mu \rangle^* A.$$

Thus the above is equivalent to $\langle \nu \rangle^* \langle \nu^{-1} \rangle^* A \sqsubseteq \langle \nu^{-1} \rangle^* \langle (\text{FCD})\mu \rangle^* A$.

And this is in turn equivalent to

$$\nu \circ \nu^{-1} \sqsubseteq \nu^{-1} \circ (\text{FCD})\mu.$$

□

DEFINITION 2069. An endofunctor μ is *normal* on endofunctor ν when $\nu \circ \nu^{-1} \sqsubseteq \nu^{-1} \circ \mu$. **FiXme: No need for ν to be endomorphism.**

OBVIOUS 2070.

- 1°. Endoreloid μ is normal on endofunctor ν iff endofunctor $(\text{FCD})\mu$ is normal on endofunctor ν .
- 2°. Endofunctor μ is normal on endoreloid ν iff endofunctor $(\text{RLD})_{\text{in}}\mu$ is normal on endofunctor ν .

COROLLARY 2071. If ν is a symmetric endofunctor and $\mu \sqsupseteq \nu^{-1}$, then it is normal.

COROLLARY 2072. (generalization of proposition 1 in [1]) If ν is a symmetric endofunctor and $\text{Compl } \mu \sqsupseteq \nu^{-1}$, then it is normal.

DEFINITION 2073. A funcoïd ν is *normally reloidazable* iff there exist a reloid μ such that (μ, ν) is normal and $\nu = \text{Compl}(\text{FCD})\mu$.

DEFINITION 2074. A funcoïd ν is *normally quasi-uniformizable* iff there exist a quasi-uniform space (= reflexive and transitive reloid) μ such that (μ, ν) is normal and $\nu = \text{Compl}(\text{FCD})\mu$.

PROPOSITION 2075. A funcoïd ν is normally reloidazable iff there exist a funcoïd μ such that μ is normal on ν and $\nu = \text{Compl } \mu$.

PROPOSITION 2076. A funcoïd ν is normally quasi-uniformizable iff there exist a quasi-proximity space (= reflexive and transitive funcoïd) μ such that μ is normal on ν and $\nu = \text{Compl } \mu$.

PROOF. Obvious 2070 and the fact that (FCD) is an isomorphism between reflexive and transitive funcoïds and reflexive and transitive reloids. \square

In other words, it is normally reloidazable or normally quasi-uniformizable when

$$(\text{Compl } \mu) \circ (\text{Compl } \mu)^{-1} \sqsubseteq (\text{Compl } \mu)^{-1} \circ \mu$$

for suitable μ .

2. Urysohn's lemma and friends

For a detailed proof of Urysohn's lemma see also:

http://homepage.math.uiowa.edu/~jsimon/COURSES/M132Fall07/UrysohnLemma_v5.pdf

https://proofwiki.org/wiki/Urysohn's_Lemma

<http://planetmath.org/prooffofurysohnslemma>

https://en.wikipedia.org/wiki/Proximity_space says that "The resulting topology is always completely regular. This can be proven by imitating the usual proofs of Urysohn's lemma, using the last property of proximal neighborhoods to create the infinite increasing chain used in proving the lemma."

Below follows an alternative proof of Urysohn lemma. Warning: This proof is conditional, based on unproved conjecture 793.

LEMMA 2077. If $\langle \mu \rangle \mathcal{A} \asymp \mathcal{B}$ for a complete funcoïd μ and \mathcal{A}, \mathcal{B} are filters on relevant sets, then there exists $U \in \text{up } \mu$ such that $\langle U \rangle \mathcal{A} \asymp \mathcal{B}$.

PROOF. Prove that $\left\{ \frac{\langle U \rangle \mathcal{A}}{U \in \text{up } \mu} \right\}$ is a filter base. That it is nonempty is obvious.

Let $\mathcal{X}, \mathcal{Y} \in \left\{ \frac{\langle U \rangle \mathcal{A}}{U \in \text{up } \mu} \right\}$. Then $\mathcal{X} = \langle U_{\mathcal{X}} \rangle \mathcal{A}, \mathcal{Y} = \langle U_{\mathcal{Y}} \rangle \mathcal{A}$. Because μ is complete, we have (proposition 1040) $U_{\mathcal{X}} \sqcap U_{\mathcal{Y}} \in \text{up } \mu$. Thus $\mathcal{X}, \mathcal{Y} \sqsupseteq \langle U_{\mathcal{X}} \sqcap U_{\mathcal{Y}} \rangle \mathcal{A} \in \left\{ \frac{\langle U \rangle \mathcal{A}}{U \in \text{up } \mu} \right\}$.

Thus $\langle \mu \rangle \mathcal{A} \asymp \mathcal{B} \Leftrightarrow \mathcal{B} \sqcap \langle \mu \rangle \mathcal{A} = \perp \Leftrightarrow \exists U \in \text{up } \mu : \mathcal{B} \sqcap \langle U \rangle \mathcal{A} = \perp \Leftrightarrow \exists U \in \text{up } \mu : \langle U \rangle \mathcal{A} \asymp \mathcal{B}$. \square

COROLLARY 2078. If $\langle \mu \rangle \mathcal{A} \asymp \langle \mu \rangle \mathcal{B}$ for a complete funcoïd μ and \mathcal{A}, \mathcal{B} are filters on relevant sets, then there exists $U \in \text{up } \mu$ such that $\langle U \rangle \mathcal{A} \asymp \langle U \rangle \mathcal{B}$.

PROOF. Applying the lemma twice we can obtain $P, Q \in \text{up } \mu$ such that $\langle P \rangle \mathcal{A} \asymp \langle Q \rangle \mathcal{B}$. But because μ is complete, we have $U = P \sqcap Q \in \text{up } \mu$, while obviously $\langle U \rangle \mathcal{A} \asymp \langle U \rangle \mathcal{B}$. \square

LEMMA 2079. (assuming conjecture 793) For every $U \in \text{up } \mu$ (where μ is a T_4 topological space) such that $\neg(A [U \circ U^{-1}]^* B)$ there is $W \in \text{up } \mu$ such that $U \circ U^{-1} \sqsupseteq W \circ W^{-1} \circ W \circ W^{-1}$. For it holds $\neg(A [W \circ W^{-1}]^* B)$. We can assume that $\langle W \rangle^* X$ is open for every set X .

PROOF. $U \circ U^{-1} \in \text{up}(\mu \circ \mu^{-1}) \subseteq \text{up}(\mu \circ \mu^{-1} \circ \mu \circ \mu^{-1})$ (normality used). Thus by the conjecture there exists $W \in \text{up} \mu$ such that $U \circ U^{-1} \supseteq W \circ W^{-1} \circ W \circ W^{-1}$. $W \circ W^{-1} \subseteq U \circ U^{-1}$ thus $\neg(A [W \circ W^{-1}]^* B)$.

To prove that $\langle W \rangle^* X$ is open for every set X , replace every $\langle W \rangle^* \{x\}$ with an open neighborhood $E \subseteq \langle W \rangle^* X$ of $\langle \mu \rangle^* \{x\}$ (and note that union of open sets is open). This new W holds all necessary properties. \square

LEMMA 2080. (assuming conjecture 793) For every $U \in \text{up} \mu$ (where μ is a T_4 topological space) such that $\neg(A [U \circ U^{-1}]^* B)$ there is $W \in \text{up} \mu$ such that $U \circ U^{-1} \supseteq \mu^{-1} \circ W \circ W^{-1} \circ W \circ W^{-1}$. For it holds $\neg(A [W \circ W^{-1}]^* B)$. We can assume that $\langle W \rangle^* X$ is open for every set X .

PROOF. Applying the previous lemma twice, we have some open $W \in \text{up} \mu$ such that

$$U \circ U^{-1} \supseteq W \circ W^{-1} \circ W \circ W^{-1} \circ W \circ W^{-1} \circ W \circ W^{-1}$$

and $\neg(A [W \circ W^{-1}]^* B)$. From this easily follows that

$$U \circ U^{-1} \supseteq \mu^{-1} \circ W \circ W^{-1} \circ W \circ W^{-1}.$$

\square

A modified proof of Urysohn's lemma follows. This proof is in part based on [1]. (I attempt to find common generalization of Urysohn's lemma and results from [1]).

$$\mathbb{Q}_2 \stackrel{\text{def}}{=} \left\{ \frac{k/2^n}{k, n \in \mathbb{N}, 0 < k < 2^n} \right\}.$$

THEOREM 2081. Urysohn's lemma (see Wikipedia) for disjoint closed sets A and B and function f on a topological space μ (considered as complete funcoid).

PROOF. (assuming conjecture 793) (used ProofWiki among other sources)

Because A and B are disjoint closed sets, we have $\langle \mu \rangle^* A \simeq \langle \mu \rangle^* B$. Thus by the corollary 2078 take $S_0 \in \text{up} \mu$ and $\neg(A [S_0 \circ S_0^{-1}]^* B)$.

We have $\mu \circ \mu^{-1} \circ \mu \circ \mu^{-1} \subseteq \mu \circ \mu^{-1}$ that is $\text{up}(\mu \circ \mu^{-1} \circ \mu \circ \mu^{-1}) \supseteq \text{up}(\mu \circ \mu^{-1})$.

Let's prove by induction: There is a sequence S of binary relations starting with S_0 such that $\neg(A [S_i \circ S_i^{-1}]^* B)$ and $S_i \circ S_i^{-1} \supseteq \mu^{-1} \circ S_{i+1} \circ S_{i+1}^{-1} \circ S_{i+1} \circ S_{i+1}^{-1}$. It directly follows from the lemma (and uses the conjecture).

Denote $U_i = S_{i+1} \circ S_{i+1}^{-1}$. We have $U_i \supseteq \mu^{-1} \circ U_{i+1} \circ U_{i+1}$ and $\neg(A [U_i]^* B)$.

By reflexivity of μ we have $U_{i+1} \subseteq U_{i+1} \circ U_{i+1} \subseteq U_i$.

Define fractional degree of U : $U^r \stackrel{\text{def}}{=} U_1^{r_1} \circ \dots \circ U_{l_r}^{r_{l_r}}$ for every $r \in \mathbb{Q}_2$ where $r_1 \dots r_{l_r}$ is the binary expansion of r .

Prove $U_r \subseteq U_0$. It is enough to prove $U_0 \supseteq U_1 \circ \dots \circ U_{l_r}$. It follows from $U_2 \circ \dots \circ U_{l_r} \subseteq U_1$, $U_3 \circ \dots \circ U_{l_r} \subseteq U_2$, \dots , $U_{l_r} \subseteq U_{l_r-1}$ what was shown above.

Let's prove: For each $p, q \in \mathbb{Q}_2$ such that $p < q$ we have $\mu^{-1} \circ U^p \subseteq U^q$. We can assume binary expansion of p and q be the same length c (add zeros at the end of the shorter one). Now it is enough to prove

$$U_k \circ U_{k+1}^{q_{k+1}} \circ \dots \circ U_c^{q_c} \supseteq \mu^{-1} \circ U_{k+1}^{p_{k+1}} \circ U_{k+2}^{p_{k+2}} \circ \dots \circ U_c^{p_c}.$$

But for this it's enough

$$U_k \supseteq \mu^{-1} \circ U_{k+1} \circ U_{k+2} \circ \dots \circ U_c$$

what can be easily proved by induction: If $k = c$ then it takes the form $U_k \supseteq \mu^{-1}$ what is obvious. Suppose it holds for k . Then $U_{k-1} \supseteq \mu^{-1} \circ U_k \circ U_k \supseteq \mu^{-1} \circ U_k \circ$

$\mu^{-1} \circ U_{k+1} \circ U_{k+2} \circ \cdots \circ U_c \supseteq \mu^{-1} \circ U_k \circ U_{k+1} \circ U_{k+2} \circ \cdots \circ U_c$, that is it holds for all natural $k \leq c$.

It is easy to prove that $\langle U^r \rangle^* X$ is open for every set X .

We have $\langle \mu^{-1} \rangle^* \langle U^p \rangle^* X \subseteq \langle U^q \rangle^* X$.

$$f(z) \stackrel{\text{def}}{=} \inf \left(\{1\} \cup \left\{ \frac{q \in \mathbb{Q}_2}{z \in \langle U^q \rangle^* A} \right\} \right).$$

f is properly defined because $\{1\} \cup \left\{ \frac{q \in \mathbb{Q}_2}{z \in \langle U^q \rangle^* A} \right\}$ is nonempty and bounded.

If $z \in A$ then $z \in \langle U^q \rangle^* A$ for every $q \in \mathbb{Q}_2$, thus $f(z) = 0$, because obviously $U^q \supseteq 1$.

If $z \in B$ then $z \notin \langle U^q \rangle^* A$ for every $q \in \mathbb{Q}_2$, thus $f(z) = 1$, because $U^q \subseteq U_0$.

It remains to prove that f is continuous.

Let $D(x) = \{1\} \cup \left\{ \frac{q \in \mathbb{Q}_2}{z \in \langle U^q \rangle^* A} \right\}$.

To show that f is continuous, we first prove two smaller results:

(a) $x \in \langle \mu^{-1} \rangle^* \langle U^r \rangle^* A \Rightarrow f(x) \leq r$.

We have $x \in \langle \mu^{-1} \rangle^* \langle U^r \rangle^* A \Rightarrow \forall s > r : x \in \langle U^s \rangle^* A$, so $D(x)$ contains all rationals greater than r . Thus $f(x) \leq r$ by definition of f .

(b) $x \notin \langle U^r \rangle^* A \Rightarrow f(x) \geq r$.

We have $x \notin \langle U^r \rangle^* A \Rightarrow \forall s < r : x \notin \langle U^s \rangle^* A$. So $D(x)$ contains no rational less than r . Thus $f(x) \geq r$.

Let $x_0 \in S$ and let $]c; d[$ be an open real interval containing $f(x_0)$. We will find a neighborhood T of x_0 such that $\langle f \rangle^* T \subseteq]c; d[$.

Choose $p, q \in \mathbb{Q}$ such that $c < p < f(x_0) < q < d$. Let $T = \langle U^q \rangle^* A \setminus \langle \mu^{-1} \rangle^* \langle U^p \rangle^* A$.

Then since $f(x_0) < q$, we have that (b) implies vacuously that $x \in \langle U^q \rangle^* A$.

Since $f(x_0) > p$, (a) implies $x_0 \notin \langle U^p \rangle^* A$.

Hence $x_0 \in T$. Then T is a neighborhood of x_0 because T is open.

Finally, let $x \in T$.

Then $x \in \langle U^q \rangle^* A \subseteq \langle \mu^{-1} \rangle^* \langle U^q \rangle^* A$. So $f(x) \leq q$ by (a).

Also $x \notin \langle \mu^{-1} \rangle^* \langle U^p \rangle^* A$, so $x \notin \langle U^p \rangle^* A$ and $f(x) \geq p$ by (b).

Thus: $f(x) \in [p; q] \subseteq]c; d[$.

Therefore f is continuous.

Claim A: $f(x) > q \Rightarrow x \notin \langle \mu^{-1} \rangle^* \langle U^q \rangle^* A$

Claim B: $f(x) < q \Rightarrow x \in \langle U^q \rangle^* A$

Proof of claim A: If $f(x) > q$ then there must be some gap between q and $D(x)$; in particular, there exists some q' such that $q < q' < f(x)$. But $q' < f(x) \Rightarrow x \notin \langle U^{q'} \rangle^* A \Rightarrow x \notin \langle \mu^{-1} \rangle^* \langle U^{q'} \rangle^* A$ (using that $\langle U^r \rangle^* X$ is open).

Proof of claim B: If $f(x) < q$ then there exists $q' \in D(x)$ such that $f(x) < q' < q$, in which case $q \in D(x)$, so $x \in \langle U^q \rangle^* A$.

To show that f is continuous, it's enough to prove that preimages of $]a; 1[$ and $]0; a[$ are open.

Suppose $f(x) \in]a; 1[$. Pick some q with $a < q < f(x)$. We claim that the open set $W = X \setminus \langle f^{-1} \rangle^* \langle U^q \rangle^* A$ is a neighborhood of x that is mapped by f into $]a; 1[$. First, by (A), $f(x) > q \Rightarrow x \in W$, so W is a neighborhood of x . If y is any point of W , then $f(y)$ must be $\geq q > a$; otherwise, if $f(y) < q$, then, by (B) $y \in \langle U^q \rangle^* A \subseteq \langle f^{-1} \rangle^* \langle U^q \rangle^* A$.

Suppose $x \in f^{-1}[0; b[$ that is $f(x) < b$ and pick q such that $f(x) < q < b$. By (B) $x \in \langle U^q \rangle^* A$. We claim that the neighborhood $\langle U^q \rangle^* A$ is mapped by f into $]0; b[$. Suppose y is any point of $\langle U^q \rangle^* A$. Then $q \in D(y)$, so $f(y) \leq q < b$. \square

THEOREM 2082. (from [1]) If μ is a normal quasi-uniformity on a topological space ν , then for any nonempty subset $A \in \text{Ob } \nu$ and entourage $U \in \text{up } \mu$ there exists a continuous function $f : \text{Ob } \nu \rightarrow [0; 1]$ such that $A \sqsubseteq \langle f^{-1} \rangle^* \{0\} \sqsubseteq \langle f^{-1} \rangle^* [0; 1] \sqsubseteq \langle \nu^{-1\circ} \rangle^* \langle \nu^{-1} \rangle^* \langle U \rangle^* A$.

PROOF. Choose inductively a sequence of entourages $(U_n)_{n=0}^\infty$ such that $U_0 = U$ and $U_{n+1} \circ U_{n+1} \sqsubseteq U_n$.

Denote $l_r = \max \left\{ \frac{n \in \mathbb{N}}{r_n = 1} \right\}$.

Define $U^r = U_{l_r}^{r_{l_r}} \circ \dots \circ U_1^{r_1}$

Prove $\langle \nu^{-1} \rangle^* \langle U^q \rangle^* A \sqsubseteq \langle \nu^{-1\circ} \rangle^* \langle \nu^{-1} \rangle^* \langle U^r \rangle^* A$ for any $q < r$ in \mathbb{Q}_2 . **FixMe:** Can be easily rewritten with the formula $\langle \nu \rangle^* \langle \nu^{-1} \rangle^* \langle U^q \rangle^* A \sqsubseteq \langle \nu^{-1} \rangle^* \langle U^r \rangle^* A$ instead. It may extend to non-complete funcoinds.

There is such l that $0 = q_l < r_l = 1$ and $q_i = r_i$ for all $i < l$.

It follows $l_q \neq l \leq l_r$.

Consider variants:

$$\begin{aligned}
l_q < l. \quad \langle \nu^{-1} \rangle^* \langle U^q \rangle^* A &\sqsubseteq \langle \nu^{-1} \rangle^* \langle U_{l_q} \circ \dots \circ U_1^{q_1 q_l} \rangle^* A = \\
&\langle \nu^{-1} \rangle^* \langle U_{l_q}^{r_{l_q}} \circ \dots \circ U_1^{r_1} \rangle^* A \sqsubseteq \langle \nu^{-1} \rangle^* \langle U_{l-1}^{r_{l-1}} \circ \dots \circ U_1^{r_1} \rangle^* A \sqsubseteq \\
&\langle \nu^{-1\circ} \rangle^* \langle \nu^{-1} \rangle^* \langle U_l^{r_l} \circ U_{l-1}^{r_{l-1}} \circ \dots \circ U_1^{r_1} \rangle^* A = \langle \nu^{-1\circ} \rangle^* \langle \nu^{-1} \rangle^* \langle U^r \rangle^* A \\
&\text{(use } U_l^{r_l} \in \text{up}(\text{FCD})\mu \text{ by theorem 992).} \\
l < l_q. \text{ Inclusions } U_k \circ U_k \sqsubseteq U_{k-1} \text{ for } l < k \leq l_q + 1 \text{ guarantee that } U_{l_q+1} \circ U_{l_q} \circ \\
&\dots \circ U_{l+1} \sqsubseteq U_l \text{ and then } \langle \nu^{-1} \rangle^* \langle U^q \rangle^* A \sqsubseteq \langle \nu^{-1} \rangle^* \langle U_{l_q}^{q_1 q} \circ \dots \circ U_1^{q_1} \rangle^* A \sqsubseteq \\
&\langle \nu^{-1\circ} \rangle^* \langle \nu^{-1} \rangle^* \langle U_{l_q+1}^{q_1 q+1} \circ U_{l_q}^{q_1 q} \circ \dots \circ U_1^{q_1} \rangle^* A = \\
&\langle \nu^{-1\circ} \rangle^* \langle \nu^{-1} \rangle^* \langle U_{l_q+1} \circ U_{l_q}^{q_1 q} \circ \dots \circ U_l^0 \circ \dots \circ U_1^{q_1} \rangle^* A \sqsubseteq \\
&\langle \nu^{-1\circ} \rangle^* \langle \nu^{-1} \rangle^* \langle U_l \circ U_{l-1}^{q_1-1} \circ \dots \circ U_1^{q_1} \rangle^* A \sqsubseteq \\
&\langle \nu^{-1\circ} \rangle^* \langle \nu^{-1} \rangle^* \langle U_l^{r_l} \circ U_{l-1}^{r_{l-1}} \circ \dots \circ U_1^{r_1} \rangle^* A \sqsubseteq \\
&\langle \nu^{-1\circ} \rangle^* \langle \nu^{-1} \rangle^* \langle U_{l_r}^{r_{l_r}} \circ \dots \circ U_1^{r_1} \rangle^* A = \langle \nu^{-1\circ} \rangle^* \langle \nu^{-1} \rangle^* \langle U^r \rangle^* A.
\end{aligned}$$

Define f by the formula $f(z) = \inf \left(\{1\} \cup \left\{ \frac{q \in \mathbb{Q}_2}{z \in \langle \nu^{-1} \rangle^* \langle U^q \rangle^* A} \right\} \right)$.

It is clear?? that $A \sqsubseteq \langle f^{-1} \rangle^* \{0\}$ and $\langle f^{-1} \rangle^* [0; 1] \sqsubseteq \bigcup_{q \in \mathbb{Q}_2} \langle \nu^{-1} \rangle^* \langle U^q \rangle^* A = \bigcup_{r \in \mathbb{Q}_2} \langle \nu^{-1\circ} \rangle^* \langle \nu^{-1} \rangle^* \langle U^r \rangle^* A \sqsubseteq \langle \nu^{-1\circ} \rangle^* \langle \nu^{-1} \rangle^* \langle U_0 \rangle^* A$.

To prove that the map $f : X \rightarrow [0; 1]$ is continuous, it suffices to check that for every real number $a \in]0; 1[$ the sets $\langle f^{-1} \rangle^* [0; a[$ and $\langle f^{-1} \rangle^*]a; 1]$ are open. This follows from the equalities

$$\langle f^{-1} \rangle^* [0; a[= \bigcup_{\mathbb{Q}_2 \ni q < a} \langle \nu^{-1\circ} \rangle^* \langle \nu^{-1} \rangle^* \langle U^q \rangle^* A \text{ and } \langle f^{-1} \rangle^*]a; 1] = \bigcup_{\mathbb{Q}_2 \ni r > a} (X \setminus \langle \nu^{-1} \rangle^* \langle U^r \rangle^* A). \quad \square$$

How the formulas for normal (T_4) topological spaces and normal quasi-uniformities are related? Maybe this works: Replacing $\nu \rightarrow \mu \circ \mu^{-1}$, $\mu \rightarrow 1$ makes $\nu \circ \nu^{-1} \sqsubseteq \nu^{-1} \circ (\text{FCD})\mu \rightarrow \mu \circ \mu^{-1} \circ \mu \circ \mu^{-1} \sqsubseteq \mu \circ \mu^{-1}$.

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