Algebraic General Topology. Volume 1 addons

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Abstract. This file contains future addons for the free e-book "Algebraic General Topology. Volume 1", which are yet not enough ripe to be included into the book.

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CHAPTER 1

About this document

This file contains future addons for the free e-book "Algebraic General Topology. Volume 1", which are yet not enough ripe to be included into the book.

Theorem (including propositions, conjectures, etc.) numbers in this document start from the last theorem number in the book plus one. Theorems references inside this document are hyperlinked, but references to theorems in the book are not hyperlinked (because PDF viewer Okular 0.20.2 does not support Backward button after clicking a cross-document reference, and thus I want to avoid clicking such links).

CHAPTER 2

Unfixed categories

FiXme: This is a draft not thoroughly checked for errors.

1. Axiomatics for unfixed morphisms

DEFINITION 2056. *Category with restricted identities* is defined axiomatically: *Restricted identity* $\mathrm{id}_X^{\mathcal{C}(A,B)}$ and *projection* $A \mapsto [A]$ are described by the axioms:

1°. $\mathcal{C}$ is a category with the set of objects $\mathfrak{Z}$;
2°. every Hom-set $\mathcal{C}(A,B)$ is a lattice;
3°. $\mathfrak{Z}$ and $\mathfrak{A}$ are lattices;
4°. $A \to [A]$ is a lattice embedding from $\mathcal{C}(A,B)$ to $\mathfrak{A}$ whenever A ranges a Hom-set $\mathcal{C}(A,B)$;
5°. $\mathrm{id}_X^{\mathcal{C}(A,B)} \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ whenever $\mathfrak{A} \ni X \sqsubseteq [A] \sqcap [B]$;
6°. $\mathrm{id}_{[A]}^{\mathcal{C}(A,A)} = 1_A^{\mathcal{C}}$;
7°. $\mathrm{id}_Y^{\mathcal{C}(B,C)} \circ \mathrm{id}_X^{\mathcal{C}(A,B)} = \mathrm{id}_{X \sqcap Y}^{\mathcal{C}(A,C)}$ whenever $\mathfrak{A} \ni X \sqsubseteq [A] \sqcap [B]$ and $\mathfrak{A} \ni Y \sqsubseteq [B] \sqcap [C]$;
8°. $\forall A \in \mathfrak{A} \exists B \in \mathfrak{Z} : A \sqsubseteq [B]$.

For a *partially ordered category with restricted identities* introduce additional axiom $X \sqsubseteq Y \Rightarrow \mathrm{id}_X^{\mathcal{C}(A,B)} \sqsubseteq \mathrm{id}_Y^{\mathcal{C}(A,B)}$.

For *dagger categories with restricted identities* introduce additional axiom $\left(\mathrm{id}_X^{\mathcal{C}(A,B)}\right)^\dagger = \mathrm{id}_X^{\mathcal{C}(B,A)}$.

DEFINITION 2057. I call a category with restricted identities *injective* when the axiom $X \neq Y \Rightarrow \mathrm{id}_X^{\mathcal{C}(A,B)} \neq \mathrm{id}_Y^{\mathcal{C}(A,B)}$ whenever $X, Y \sqsubseteq [A] \sqcap [B]$ holds.

DEFINITION 2058. Define $\mathcal{E}_{\mathcal{C}}^{A,B} = \mathrm{id}_{[A] \sqcap [B]}^{\mathcal{C}(A,B)}$.

PROPOSITION 2059.

1°. If $[A] \sqsubseteq [B]$ then $\mathcal{E}_{\mathcal{C}}^{A,B}$ is a monomorphism.
2°. If $[A] \sqsupseteq [B]$ then $\mathcal{E}_{\mathcal{C}}^{A,B}$ is an epimorphism.

PROOF. We'll prove only the first as the second is dual.

Let $\mathcal{E}_{\mathcal{C}}^{A,B} \circ f = \mathcal{E}_{\mathcal{C}}^{A,B} \circ g$. Then $\mathcal{E}_{\mathcal{C}}^{B,A} \circ \mathcal{E}_{\mathcal{C}}^{A,B} \circ f = \mathcal{E}_{\mathcal{C}}^{B,A} \circ \mathcal{E}_{\mathcal{C}}^{A,B} \circ g$; $1^A \circ f = 1^A \circ g$; $f = g$. □

PROPOSITION 2060. $\mathcal{E}_{\mathcal{C}}^{B,C} \circ \mathcal{E}_{\mathcal{C}}^{A,B} = \mathcal{E}_{\mathcal{C}}^{A,C}$ if $B \sqsupseteq A \sqcap C$ (for every sets A, B, C).

PROOF. $\mathcal{E}_{\mathcal{C}}^{B,C} \circ \mathcal{E}_{\mathcal{C}}^{A,B} = \mathcal{E}_{\mathcal{C}}^{A,C}$ is equivalent to:
$\mathrm{id}_{B \sqcap C}^{\mathcal{C}(B,C)} \circ \mathrm{id}_{A \sqcap B}^{\mathcal{C}(A,B)} = \mathrm{id}_{A \sqcap C}^{\mathcal{C}(A,C)}$ what is obviously true. □

2. Rectangular embedding-restriction

DEFINITION 2061. $\iota_{B_0,B_1} f = \mathcal{E}_{\mathcal{C}}^{\mathrm{Dst}\, f, B_1} \circ f \circ \mathcal{E}_{\mathcal{C}}^{B_0, \mathrm{Src}\, f}$ for $f \in \mathrm{Hom}_{\mathcal{C}}(A_0, A_1)$.

For brevity $\iota_B f = \iota_{B,B} f$.

OBVIOUS 2062. $\iota_{B_0,B_1} f \sqsubseteq f$.

PROPOSITION 2063. $\iota_{\operatorname{Src} f,\operatorname{Dst} f} f = f$.

PROOF. $\iota_{\operatorname{Src} f,\operatorname{Dst} f} f = \mathcal{E}_{\mathcal{C}}^{\operatorname{Dst} f,\operatorname{Dst} f} \circ f \circ \mathcal{E}_{\mathcal{C}}^{\operatorname{Src} f,\operatorname{Src} f} = 1_{\mathcal{C}}^{\operatorname{Dst} f} \circ f \circ 1_{\mathcal{C}}^{\operatorname{Src} f} = f$. □

PROPOSITION 2064. The function $\iota_{B_0,B_1}|_{f\in\operatorname{Hom}_{\mathcal{C}}(A_0,A_1)}$ is injective, provided that $A_0 \sqsubseteq B_0$ and $A_1 \sqsubseteq B_1$.

PROOF. Because $\mathcal{E}_{\mathcal{C}}^{A_1,B_1}$ is a monomorphism and $\mathcal{E}_{\mathcal{C}}^{A_0,B_0}$ is an epimorphism. □

COROLLARY 2065. The function $\iota_{B_0,B_1}|_{f\in\operatorname{Hom}_{\mathcal{C}}(A_0,A_1)}$ is order embedding if $A_0 \sqsubseteq B_0 \wedge A_1 \sqsubseteq B_1$ for ordered categories with restricted identities.

3. Image and domain

Let define that $\mathscr{S}\mathcal{A} = \left\{ \frac{K\in\mathfrak{Z}}{\exists X\in\mathcal{A}:X\subseteq K} \right\}$ holds not only for filters but for any set $\mathcal{A}$ of sets.

OBVIOUS 2066. $\mathscr{S}\mathcal{A} \supseteq \mathcal{A}$.

DEFINITION 2067.
1°. $\operatorname{IM} f = \left\{ \frac{Y\in\mathfrak{Z}}{\mathcal{E}_{\mathcal{C}}^{Y,\operatorname{Dst} f}\circ\mathcal{E}_{\mathcal{C}}^{\operatorname{Dst} f,Y}\circ f=f} \right\} = \left\{ \frac{Y\in\mathfrak{Z}}{\operatorname{id}^{\mathcal{C}(\operatorname{Dst} f,\operatorname{Dst} f)}_{[Y]\sqcap[\operatorname{Dst} f]}\circ f=f} \right\}$;
2°. $\operatorname{DOM} f = \left\{ \frac{X\in\mathfrak{Z}}{f\circ\mathcal{E}_{\mathcal{C}}^{\operatorname{Src} f,X}\circ\mathcal{E}_{\mathcal{C}}^{X,\operatorname{Src} f}=f} \right\} = \left\{ \frac{X\in\mathfrak{Z}}{f\circ\operatorname{id}^{\mathcal{C}(\operatorname{Src} f,\operatorname{Src} f)}_{[X]\sqcap[\operatorname{Src} f]}=f} \right\}$.

DEFINITION 2068.
1°. $\operatorname{Im} f = \left\{ \frac{Y\in\operatorname{IM} f}{Y\sqsubseteq\operatorname{Dst} f} \right\}$;
2°. $\operatorname{Dom} f = \left\{ \frac{X\in\operatorname{DOM} f}{X\sqsubseteq\operatorname{Src} f} \right\}$.

PROPOSITION 2069.
1°. $\operatorname{IM} f = \mathscr{S}\operatorname{Im} f$;
2°. $\operatorname{DOM} f = \mathscr{S}\operatorname{Dom} f$;
3°. $\operatorname{Im} f = \langle \operatorname{Dst} f\cap\rangle^* \operatorname{IM} f$;
4°. $\operatorname{Dom} f = \langle \operatorname{Dst} f\cap\rangle^* \operatorname{DOM} f$.

PROOF. $\operatorname{IM} f = \left\{ \frac{Y\in\mathfrak{Z}}{\operatorname{id}^{\mathcal{C}(\operatorname{Dst} f,\operatorname{Dst} f)}_{[Y]\sqcap[\operatorname{Dst} f]}\circ f=f} \right\}$.
Suppose $Y \in \operatorname{IM} f$. Then take $Y' = Y \sqcap \operatorname{Dst} f$. We have $Y \sqsupseteq Y'$ and $Y' \in \operatorname{Im} f$. So $Y \in \mathscr{S}\operatorname{Im} f$. If $Y \in \mathscr{S}\operatorname{Im} f$ then $Y \in \operatorname{IM} f$ obviously. So $\operatorname{IM} f = \mathscr{S}\operatorname{Im} f$.
$\langle \operatorname{Dst} f\cap\rangle^* \operatorname{IM} f \subseteq \operatorname{Im} f$ is obvious. If $\operatorname{Im} f \subseteq \langle \operatorname{Dst} f\cap\rangle^* \operatorname{IM} f$ is also obvious.
The rest follows from symmetry. □

CONJECTURE 2070. $\operatorname{Im} f$ may be not a filter for an injective category with restricted morphisms.

PROPOSITION 2071. $\operatorname{Dst} f \in \operatorname{Im} f$; $\operatorname{Src} f \in \operatorname{Dom} f$ for every morphism f of a category with restricted identities.

PROOF. Prove $\operatorname{Dst} f \in \operatorname{Im} f$ (the other is similar): We need to prove that $\mathcal{E}_{\mathcal{C}}^{\operatorname{Dst} f,\operatorname{Dst} f}\circ\mathcal{E}_{\mathcal{C}}^{\operatorname{Dst} f,\operatorname{Dst} f}\circ f = f$ what follows from $\mathcal{E}_{\mathcal{C}}^{\operatorname{Dst} f,\operatorname{Dst} f}\circ\mathcal{E}_{\mathcal{C}}^{\operatorname{Dst} f,\operatorname{Dst} f} = 1^{\operatorname{Dst} f}$. □

PROPOSITION 2072. $\operatorname{IM} f$, $\operatorname{Im} f$, $\operatorname{DOM} f$, $\operatorname{Dom} f$ are upper sets.

PROOF. For $\operatorname{Im} f$, $\operatorname{Dom} f$ it follows from the previous proposition.
For $\operatorname{IM} f$, $\operatorname{DOM} f$ it follows from the thesis for $\operatorname{Im} f$, $\operatorname{Dom} f$. □

DEFINITION 2073.

1°. An ordered category with restricted identities is *with ordered image* iff $f \sqsubseteq g \Rightarrow \operatorname{IM} f \subseteq \operatorname{IM} g$.
2°. An ordered category with restricted identities is *with ordered domain* iff $f \sqsubseteq g \Rightarrow \operatorname{DOM} f \subseteq \operatorname{DOM} g$.
3°. An ordered category with restricted identities is *with ordered domain and image* iff it is both with ordered domain and with ordered image.

Obvious 2074.

1°. An ordered category with restricted identities is with ordered image iff $f \sqsubseteq g \Rightarrow \operatorname{Im} f \subseteq \operatorname{Im} g$.
2°. An ordered category with restricted identities is with ordered domain iff $f \sqsubseteq g \Rightarrow \operatorname{Dom} f \subseteq \operatorname{Dom} g$.
3°. An ordered category with restricted identities is with ordered domain and image iff it is both with ordered domain and with ordered image.

Obvious 2075.

1°. For an ordered category $\mathcal{C}$ with restricted identities to be with ordered image it's enough that $\mathrm{id}^{\mathcal{C}(\operatorname{Dst} f, \operatorname{Dst} f)}_{[X]} \circ f = f \wedge g \sqsubseteq f \Rightarrow \mathrm{id}^{\mathcal{C}(\operatorname{Dst} f, \operatorname{Dst} f)}_{[X]} \circ g = g$ for every parallel morphisms f and g and $\mathfrak{Z} \ni X \sqsubseteq \operatorname{Dst} f$.
2°. For an ordered category $\mathcal{C}$ with restricted identities to be with ordered domain it's enough that $f \circ \mathrm{id}^{\mathcal{C}(\operatorname{Src} f, \operatorname{Src} f)}_{[X]} = f \wedge g \sqsubseteq f \Rightarrow g \circ \mathrm{id}^{\mathcal{C}(\operatorname{Src} f, \operatorname{Src} f)}_{[X]} = g$ for every parallel morphisms f and g and $\mathfrak{Z} \ni X \sqsubseteq \operatorname{Src} f$.

Conjecture 2076. There exists a category with restricted identities which is not with ordered image.

Obvious 2077. For an ordered category with restricted identities with ordered domain and image we have $\iota_{\operatorname{Src} f, \operatorname{Dst} f} \iota_{A,B} f = f \wedge g \sqsubseteq f \Rightarrow \iota_{\operatorname{Src} f, \operatorname{Dst} f} \iota_{A,B} g = g$ for parallel morphisms f and g.

Definition 2078.

1°. $\operatorname{im} f = \min \operatorname{Im} f$;
2°. $\operatorname{dom} f = \min \operatorname{Dom} f$.

Note 2079. It seems that im and dom are defined not for every category with restricted identities.

Proposition 2080.

1°. $\operatorname{im} f = \min \operatorname{IM} f$;
2°. $\operatorname{dom} f = \min \operatorname{DOM} f$.

Proof. It follows from $\operatorname{IM} f = \mathscr{S} \operatorname{Im} f$ (and likewise for $\operatorname{dom} f$). □

Theorem 2081. $\operatorname{DOM}(g \circ f) \supseteq \operatorname{DOM} f$, $\operatorname{IM}(g \circ f) \supseteq \operatorname{IM} g$, $\operatorname{Dom}(g \circ f) \supseteq \operatorname{Dom} f$, $\operatorname{Im}(g \circ f) \supseteq \operatorname{Im} g$.

Proof. $\mathcal{E}^{Y, \operatorname{Dst} f}_{\mathcal{C}} \circ \mathcal{E}^{\operatorname{Dst} f, Y}_{\mathcal{C}} \circ g \circ f = g \circ f \Leftarrow \mathcal{E}^{Y, \operatorname{Dst} f}_{\mathcal{C}} \circ \mathcal{E}^{\operatorname{Dst} f, Y}_{\mathcal{C}} \circ g = g$ and it implies $\operatorname{IM}(g \circ f) \supseteq \operatorname{IM} g$. The rest follows easily. □

Corollary 2082. $\operatorname{dom}(g \circ f) \sqsubseteq \operatorname{dom} f$, $\operatorname{im}(g \circ f) \sqsubseteq \operatorname{im} g$ whenever dom/im are defined.

4. Equivalent morphisms

Proposition 2083. $\iota_{A,B} \iota_{X,Y} f = \iota_{A,B} f$ for every sets A, B, X, Y whenever $\operatorname{DOM} f$ and $\operatorname{IM} f$ are filters and $X \in \operatorname{DOM} f$, $Y \in \operatorname{IM} f$.

PROOF. $\iota_{A,B}f = \mathcal{E}_{\mathcal{C}}^{\mathrm{Dst}\, f,B} \circ f \circ \mathcal{E}_{\mathcal{C}}^{A,\mathrm{Src}\, f} =$ (by definition of $\mathrm{IM}\, f$ and $\mathrm{DOM}\, f$) $= \mathcal{E}_{\mathcal{C}}^{\mathrm{Dst}\, f,B} \circ \mathcal{E}_{\mathcal{C}}^{Y,\mathrm{Dst}\, f} \circ \mathcal{E}_{\mathcal{C}}^{\mathrm{Dst}\, f,Y} \circ f \circ \mathcal{E}_{\mathcal{C}}^{X,\mathrm{Src}\, f} \circ \mathcal{E}_{\mathcal{C}}^{\mathrm{Src}\, f,X} \circ \mathcal{E}_{\mathcal{C}}^{A,\mathrm{Src}\, f} = \mathcal{E}_{\mathcal{C}}^{Y,B} \circ \mathcal{E}_{\mathcal{C}}^{\mathrm{Dst}\, f,Y} \circ f \circ \mathcal{E}_{\mathcal{C}}^{X,\mathrm{Src}\, f} \circ \mathcal{E}_{\mathcal{C}}^{A,X} = \iota_{A,B}\iota_{X,Y}f$

because $\mathcal{E}^{\mathrm{Dst}\, f,B} \circ \mathcal{E}^{Y,\mathrm{Dst}\, f} \circ \mathcal{E}^{\mathrm{Dst}\, f,Y} = \mathrm{id}^{\mathcal{C}(\mathrm{Dst}\, f,B)}_{Y \sqcap \mathrm{Dst}\, f \sqcap B} = \mathrm{id}^{\mathcal{C}(Y,B)}_{Y \sqcap B} \circ \mathrm{id}^{\mathcal{C}(\mathrm{Dst}\, f,Y)}_{Y \sqcap \mathrm{Dst}\, f} = \mathcal{E}^{Y,B} \circ \mathcal{E}^{\mathrm{Dst}\, f,Y}$ and thus $\mathcal{E}_{\mathcal{C}}^{\mathrm{Dst}\, f,B} \circ \mathcal{E}_{\mathcal{C}}^{Y,\mathrm{Dst}\, f} \circ \mathcal{E}_{\mathcal{C}}^{\mathrm{Dst}\, f,Y} = \mathcal{E}_{\mathcal{C}}^{Y,B} \circ \mathcal{E}_{\mathcal{C}}^{\mathrm{Dst}\, f,Y}$ and similarly for $\mathcal{E}_{\mathcal{C}}^{X,\mathrm{Src}\, f} \circ \mathcal{E}_{\mathcal{C}}^{\mathrm{Src}\, f,X} \circ \mathcal{E}_{\mathcal{C}}^{A,\mathrm{Src}\, f}$. □

DEFINITION 2084. I call two morphisms $f \in \mathcal{C}(A_0, B_0)$ and $g \in \mathcal{C}(A_1, B_1)$ of a category with restricted morphisms *equivalent* (and denote $f \sim g$) when

$$\iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f = \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} g.$$

PROPOSITION 2085. $f \sim g$ iff $\iota_{A,B}f = \iota_{A,B}g$ for some $A \in \mathrm{DOM}\, f \cap \mathrm{DOM}\, g$, $B \in \mathrm{IM}\, f \cap \mathrm{IM}\, g$.

PROOF. Both

$$\iota_{A,B}f = \iota_{A,B}g \Rightarrow \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f = \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} g$$

and

$$\iota_{A,B}f = \iota_{A,B}g \Leftarrow \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f = \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} g$$

follow from proposition 2083. □

THEOREM 2086. Let $f : A_0 \to B_0$ and $g : A_1 \to B_1$ (for a partially ordered category with restricted identities). The following are pairwise equivalent:

1°. $f \sim g$;
2°. $\iota_{A_1,B_1} f = g$ and $\iota_{A_0,B_0} g = f$;
3°. $\iota_{A_1,B_1} f \sqsupseteq g$ and $\iota_{A_0,B_0} g \sqsupseteq f$.

PROOF.

1°⇒2°. $\iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f = \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} g$; $\iota_{A_1,B_1}\iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f = \iota_{A_1,B_1}\iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} g$; $\iota_{A_1,B_1} f = \iota_{A_1,B_1} g$; $\iota_{A_1,B_1} f = g$. $\iota_{A_0,B_0} g = f$ is similar.

3°⇒1°. Let $\iota_{A_1,B_1} f \sqsupseteq g$ and $\iota_{A_0,B_0} g \sqsupseteq f$.
$\iota_{A_1,B_1}\iota_{A_0,B_0} g \sqsupseteq g$;
$\mathcal{E}^{B_0,B_1} \circ \mathcal{E}^{B_1,B_0} \circ g \circ \mathcal{E}^{A_0,A_1} \circ \mathcal{E}^{A_1,A_0} \sqsupseteq g$;
$\mathrm{id}^{\mathcal{C}(B_1,B_1)}_{[B_0] \sqcap [B_1]} \circ g \circ \mathrm{id}^{\mathcal{C}(A_1,A_1)}_{[A_0] \sqcap [A_1]} \sqsupseteq g$; $\mathrm{id}^{\mathcal{C}(B_1,B_1)}_{[B_0] \sqcap [B_1]} \circ g \sqsupseteq g$; $\mathrm{id}^{\mathcal{C}(B_1,B_1)}_{[B_0] \sqcap [B_1]} \circ g = g$;
$\mathrm{id}^{\mathcal{C}(B_0 \sqcap B_1,B_1)}_{[B_0] \sqcap [B_1]} \circ \mathrm{id}^{\mathcal{C}(B_1,B_0 \sqcap B_1)}_{[B_0] \sqcap [B_1]} \circ g = g$; $\mathcal{E}^{B_0 \sqcap B_1, B_1} \circ \mathcal{E}^{B_1, B_0 \sqcap B_1} \circ g = g$.
Thus $B_0 \sqcap B_1 \in \mathrm{Im}\, g$. Similarly $A_0 \sqcap A_1 \in \mathrm{Dom}\, g$.
So $\iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f = \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} \iota_{A_0,B_0} g = \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} g$.

2°⇒3°. Obvious.

□

PROPOSITION 2087. Above defined equivalence of morphisms (for a small category) is an equivalence relation.

PROOF.

Reflexivity. Obvious.
Symmetry. Obvious.
Transitivity. Let $f \sim g$ and $g \sim h$ for $f : A_0 \to B_0$, $g : A_1 \to B_1$, $h : A_2 \to B_2$. Then $\iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f = \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} g$ and $\iota_{A_1 \sqcup A_2, B_1 \sqcup B_2} g = \iota_{A_1 \sqcup A_2, B_1 \sqcup B_2} h$.
Thus

$$\iota_{A_0 \sqcup A_1 \sqcup A_2, B_0 \sqcup B_1 \sqcup B_2}\iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f = \iota_{A_0 \sqcup A_1 \sqcup A_2, B_0 \sqcup B_1 \sqcup B_2}\iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} g$$

and

$$\iota_{A_0\sqcup A_1\sqcup A_2,B_0\sqcup B_1\sqcup B_2}\iota_{A_1\sqcup A_2,B_1\sqcup B_2}g = \iota_{A_0\sqcup A_1\sqcup A_2,B_0\sqcup B_1\sqcup B_2}\iota_{A_1\sqcup A_2,B_1\sqcup B_2}h$$

that is (proposition 2083)

$$\iota_{A_0\sqcup A_1\sqcup A_2,B_0\sqcup B_1\sqcup B_2}f = \iota_{A_0\sqcup A_1\sqcup A_2,B_0\sqcup B_1\sqcup B_2}g$$

and

$$\iota_{A_0\sqcup A_1\sqcup A_2,B_0\sqcup B_1\sqcup B_2}g = \iota_{A_0\sqcup A_1\sqcup A_2,B_0\sqcup B_1\sqcup B_2}h.$$

Combining, $\iota_{A_0\sqcup A_1\sqcup A_2,B_0\sqcup B_1\sqcup B_2}f = \iota_{A_0\sqcup A_1\sqcup A_2,B_0\sqcup B_1\sqcup B_2}h$ and thus

$$\iota_{A_0\sqcup A_2,B_0\sqcup B_2}\iota_{A_0\sqcup A_1\sqcup A_2,B_0\sqcup B_1\sqcup B_2}f = \iota_{A_0\sqcup A_2,B_0\sqcup B_2}\iota_{A_0\sqcup A_1\sqcup A_2,B_0\sqcup B_1\sqcup B_2}h;$$

(again proposition 2083) $\iota_{A_0\sqcup A_2,B_0\sqcup B_2}f = \iota_{A_0\sqcup A_2,B_0\sqcup B_2}h$ that is $f \sim h$. □

PROPOSITION 2088. $[f] = \left\{ \frac{\iota_{A,B}f}{A\in\mathrm{DOM}\,f, B\in\mathrm{IM}\,f} \right\}$.

PROOF. If $A \in \mathrm{DOM}\,f$, $B \in \mathrm{IM}\,f$ then

$$\iota_{A\sqcup\mathrm{Src}\,f,B\sqcup\mathrm{Dst}\,f}\iota_{A,B}f = \iota_{A\sqcup\mathrm{Src}\,f,B\sqcup\mathrm{Dst}\,f}f.$$

Thus $\iota_{A,B}f \sim f$ that is $\iota_{A,B}f \in [f]$.

Let now $g \in [f]$ that is $f \sim g$;

$$\iota_{\mathrm{Src}\,f\sqcup\mathrm{Src}\,g,\mathrm{Dst}\,f\sqcup\mathrm{Dst}\,g}f = \iota_{\mathrm{Src}\,f\sqcup\mathrm{Src}\,g,\mathrm{Dst}\,f\sqcup\mathrm{Dst}\,g}g.$$

Take $A = \mathrm{Src}\,g$, $B = \mathrm{Dst}\,g$. We have

$$\iota_{A,B}\iota_{\mathrm{Src}\,f\sqcup\mathrm{Src}\,g,\mathrm{Dst}\,f\sqcup\mathrm{Dst}\,g}f = \iota_{A,B}\iota_{\mathrm{Src}\,f\sqcup\mathrm{Src}\,g,\mathrm{Dst}\,f\sqcup\mathrm{Dst}\,g}g;$$
$$\iota_{A,B}f = \iota_{A,B}g = g.$$

□

PROPOSITION 2089.

1°. $\mathrm{IM}\,f = \left\{ \frac{Y\in\mathfrak{Z}}{\mathcal{E}_{\mathcal{C}}^{\mathrm{Dst}\,f,Y}\circ f\sim f} \right\}$;

2°. $\mathrm{DOM}\,f = \left\{ \frac{X\in\mathfrak{Z}}{f\circ\mathcal{E}_{\mathcal{C}}^{X,\mathrm{Src}\,f}\sim f} \right\}$.

PROOF.

$$\mathcal{E}_{\mathcal{C}}^{\mathrm{Dst}\,f,Y}\circ f \sim f \Leftrightarrow \iota_{\mathrm{Src}\,f,Y\sqcup\mathrm{Dst}\,f}(\mathcal{E}_{\mathcal{C}}^{\mathrm{Dst}\,f,Y}\circ f) = \iota_{\mathrm{Src}\,f,Y\sqcup\mathrm{Dst}\,f}f \Leftrightarrow$$
$$\mathcal{E}^{Y,Y\sqcup\mathrm{Dst}\,f}\circ\mathcal{E}^{\mathrm{Dst}\,f,Y}\circ f\circ\mathcal{E}^{\mathrm{Src}\,f,\mathrm{Src}\,f} = \mathcal{E}^{\mathrm{Dst}\,f,Y\sqcup\mathrm{Dst}\,f}\circ f\circ\mathcal{E}^{\mathrm{Src}\,f,\mathrm{Src}\,f} \Leftrightarrow$$
$$\mathcal{E}^{Y,Y\sqcup\mathrm{Dst}\,f}\circ\mathcal{E}^{\mathrm{Dst}\,f,Y}\circ f = \mathcal{E}^{\mathrm{Dst}\,f,Y\sqcup\mathrm{Dst}\,f}\circ f \Leftrightarrow \text{(proposition 2059)}$$
$$\Leftrightarrow \mathcal{E}^{Y\sqcup\mathrm{Dst}\,f,\mathrm{Dst}\,f}\circ\mathcal{E}^{Y,Y\sqcup\mathrm{Dst}\,f}\circ\mathcal{E}^{\mathrm{Dst}\,f,Y}\circ f = \mathcal{E}^{Y\sqcup\mathrm{Dst}\,f,\mathrm{Dst}\,f}\circ\mathcal{E}^{\mathrm{Dst}\,f,Y\sqcup\mathrm{Dst}\,f}\circ f \Leftrightarrow$$
$$\mathcal{E}^{Y,\mathrm{Dst}\,f}\circ\mathcal{E}^{\mathrm{Dst}\,f,Y}\circ f = f.$$

From this our thesis follows obviously. □

PROPOSITION 2090. $\iota_{A_1,B_1}\iota_{A_0,B_0}f \sqsubseteq \iota_{A_1,B_1}f$.

PROOF. $\iota_{A_1,B_1}\iota_{A_0,B_0}f = \mathcal{E}^{B_0,B_1}\circ\mathcal{E}^{\mathrm{Dst}\,f,B_0}\circ f\circ\mathcal{E}^{A_0,\mathrm{Src}\,f}\circ\mathcal{E}^{A_1,A_0} = \mathrm{id}^{\mathcal{C}(B_0,B_1)}_{[B_0]\sqcap[B_1]}\circ\mathrm{id}^{\mathcal{C}(\mathrm{Dst}\,f,B_0)}_{[\mathrm{Dst}\,f]\sqcap[B_0]}\circ f\circ\mathrm{id}^{\mathcal{C}(A_0,\mathrm{Src}\,f)}_{[A_0]\sqcap[\mathrm{Src}\,f]}\circ\mathrm{id}^{\mathcal{C}(A_1,A_0)}_{[A_1]\sqcap[A_0]} = \mathrm{id}^{\mathcal{C}(\mathrm{Dst}\,f,B_1)}_{[\mathrm{Dst}\,f]\sqcap[B_0]\sqcap[B_1]}\circ f\circ\mathrm{id}^{\mathcal{C}(A_1,\mathrm{Src}\,f)}_{[A_0]\sqcap[A_1]\sqcap[\mathrm{Src}\,f]} \sqsubseteq \mathrm{id}^{\mathcal{C}(\mathrm{Dst}\,f,B_1)}_{[\mathrm{Dst}\,f]\sqcap[B_1]}\circ f\circ\mathrm{id}^{\mathcal{C}(A_1,\mathrm{Src}\,f)}_{[A_1]\sqcap[\mathrm{Src}\,f]} = \iota_{A_1,B_1}f.$ □

5. Binary product

DEFINITION 2091. The category *with binary product morphism* is a category with restricted identities and additional axioms

1°. $\mathrm{id}_Y^{\mathcal{C}(B,B)} \circ f \circ \mathrm{id}_X^{\mathcal{C}(A,A)} = f \sqcap (X \times_{A,B} Y)$ (holding for every $A, B \in \mathfrak{Z}$, $\mathfrak{A} \ni X \sqsubseteq [A]$, $\mathfrak{A} \ni Y \sqsubseteq [B]$, $X \times_{A,B} Y \in \mathcal{C}(A,B)$ and morphism $f \in \mathcal{C}(A,B)$);
2°. $\iota_{A_1,B_1}(X \times_{A_0,B_0} Y) = X \times_{A_1,B_1} Y$ whenever $X \sqsubseteq [A_0] \sqcap [A_1]$ and $Y \sqsubseteq [B_0] \sqcap [B_1]$.

PROPOSITION 2092. The second axiom is equivalent to the following axiom:

1°. $f \sim X \times_{A_0,B_0} Y \Leftrightarrow f = X \times_{A_1,B_1} Y$ whenever $X \sqsubseteq [A_0] \sqcap [A_1]$ and $Y \sqsubseteq [B_0] \sqcap [B_1]$, $f : A_1 \to B_1$.

PROOF.

$\Leftarrow$. Obvious.

$\Rightarrow$. $f \sim X \times_{A_0,B_0} Y \Leftarrow f = X \times_{A_1,B_1} Y$ because $\iota_{A_1,B_1}(X \times_{A_0,B_0} Y) = X \times_{A_1,B_1} Y$ and $\iota_{A_0,B_0}(X \times_{A_1,B_1} Y) = X \times_{A_0,B_0} Y$.

Let's prove $f \sim X \times_{A_0,B_0} Y \Rightarrow f = X \times_{A_1,B_1} Y$. Really, if $f \sim X \times_{A_0,B_0} Y$ then $f = \iota_{A_1,B_1} f \sim \iota_{A_1,B_1}(X \times_{A_0,B_0} Y) = X \times_{A_1,B_1} Y$ and thus $f = X \times_{A_1,B_1} Y$. □

PROPOSITION 2093. $[A] \times_{A,B} [B]$ is the greatest morphism $\top^{\mathcal{C}(A,B)} : A \to B$.

PROOF. It's enough to prove $f \sqcap ([A] \times_{A,B} [B]) = f$ for every $f : A \to B$. Really, $f \sqcap ([A] \times_{A,B} [B]) = \mathrm{id}_B^{\mathcal{C}(B,B)} \circ f \circ \mathrm{id}_A^{\mathcal{C}(A,A)} = 1^B \circ f \circ 1^A = f$. □

PROPOSITION 2094. For every category with binary product morphism

$$X \times_{A,B} Y = \mathrm{id}_Y^{\mathcal{C}(B,B)} \circ \top^{\mathcal{C}(A,B)} \circ \mathrm{id}_X^{\mathcal{C}(A,A)}$$

PROOF. $X \times_{A,B} Y \sqsupseteq \mathrm{id}_Y^{\mathcal{C}(B,B)} \circ \top^{\mathcal{C}(A,B)} \circ \mathrm{id}_X^{\mathcal{C}(A,A)}$ because $\mathrm{id}_Y^{\mathcal{C}(B,B)} \circ \top^{\mathcal{C}(A,B)} \circ \mathrm{id}_X^{\mathcal{C}(A,A)} = \top^{\mathcal{C}(A,B)} \sqcap (X \times_{A,B} Y)$.

$\mathrm{id}_Y^{\mathcal{C}(B,B)} \circ \top^{\mathcal{C}(A,B)} \circ \mathrm{id}_X^{\mathcal{C}(A,A)} \sqsupseteq \mathrm{id}_Y^{\mathcal{C}(B,B)} \circ (X \times_{A,B} Y) \circ \mathrm{id}_X^{\mathcal{C}(A,A)} = (X \times_{A,B} Y) \sqcap (X \times_{A,B} Y) = X \times_{A,B} Y$. □

PROPOSITION 2095. $\iota_{A,B}(f \sqcap g) = \iota_{A,B} f \sqcap \iota_{A,B} g$ for every parallel morphisms f and g and objects A and B, whenever all $\mathcal{E}^{X,Y}$ are metamonovalued and metainjective.

PROOF. $\iota_{A,B}(f \sqcap g) = \mathcal{E}^{\mathrm{Dst}\, f,B} \circ (f \sqcap g) \circ \mathcal{E}^{A,\mathrm{Src}\, f} = (\mathcal{E}^{\mathrm{Dst}\, f,B} \circ f \circ \mathcal{E}^{A,\mathrm{Src}\, f}) \sqcap (\mathcal{E}^{\mathrm{Dst}\, f,B} \circ g \circ \mathcal{E}^{A,\mathrm{Src}\, f}) = \iota_{A,B} f \sqcap \iota_{A,B} g$. □

PROPOSITION 2096. $(X_0 \times_{A,B} Y_0) \sqcap (X_1 \times_{A,B} Y_1) = (X_0 \sqcap X_1) \times_{A,B} (Y_0 \sqcap Y_1)$.

PROOF. $(X_0 \times_{A,B} Y_0) \sqcap (X_1 \times_{A,B} Y_1) = \mathrm{id}_{Y_1}^{\mathcal{C}(B,B)} \circ (X_0 \times_{A,B} Y_0) \circ \mathrm{id}_{X_1}^{\mathcal{C}(A,A)} = \mathrm{id}_{Y_1}^{\mathcal{C}(B,B)} \circ \mathrm{id}_{Y_0}^{\mathcal{C}(B,B)} \circ \top^{\mathcal{C}(A,B)} \circ \mathrm{id}_{X_1}^{\mathcal{C}(A,A)} \circ \mathrm{id}_{X_0}^{\mathcal{C}(A,A)} = \mathrm{id}_{Y_0 \sqcap Y_1}^{\mathcal{C}(B,B)} \circ \top^{\mathcal{C}(A,B)} \circ \mathrm{id}_{X_0 \sqcap X_1}^{\mathcal{C}(A,A)} = (X_0 \sqcap X_1) \times_{A,B} (Y_0 \sqcap Y_1)$. □

PROPOSITION 2097. For a category with binary product morphism $\mathrm{Im}\, f$, $\mathrm{Dom}\, f$, $\mathrm{IM}\, f$, and $\mathrm{DOM}\, f$ are filters.

PROOF. That they are upper sets was proved above.

To prove that $\mathrm{Im}\, f$ is a filter it remains to show $A, B \in \mathrm{Im}\, f \Leftrightarrow A \sqcap B \in \mathrm{Im}\, f$. Really, $A, B \in \mathrm{Im}\, f \Leftrightarrow \top \times A \sqsupseteq f \wedge \top \times B \sqsupseteq f \Rightarrow \top \times (A \sqcap B) \sqsupseteq f \Leftrightarrow A \sqcap B \in \mathrm{Im}\, f$.

$\mathrm{Dom}\, f$ is similar.

The thesis for $\mathrm{IM}\, f$, $\mathrm{DOM}\, f$ follows from above proved for $\mathrm{Im}\, f$, $\mathrm{Dom}\, f$. □

NOTE 2098. For example for below defined category of funcoids (with binary product morphism), these filters are filters on filters on sets not filters of sets and thus are not the same as im and dom.

6. Operations on the set of unfixed morphisms

6.1. Semigroup of unfixed morphisms.

PROPOSITION 2099. Let $f : A_0 \to A_1$ and $g : A_1 \to A_2$ and $A_1 \sqsubseteq B_1$. Then $\iota_{B_0,B_2}(g \circ f) = \iota_{B_1,B_2} g \circ \iota_{B_0,B_1} f$.

PROOF. $\iota_{B_0,B_2}(g \circ f) = \mathcal{E}_{\mathcal{C}}^{A_2,B_2} \circ g \circ f \circ \mathcal{E}_{\mathcal{C}}^{B_0,A_0} = \mathcal{E}_{\mathcal{C}}^{A_2,B_2} \circ g \circ 1^{A_1} \circ f \circ \mathcal{E}_{\mathcal{C}}^{B_0,A_0} = \mathcal{E}_{\mathcal{C}}^{A_2,B_2} \circ g \circ \mathrm{id}_{A_1}^{\mathcal{C}(\operatorname{Dst} f, \operatorname{Src} g)} \circ f \circ \mathcal{E}_{\mathcal{C}}^{B_0,A_0} = \mathcal{E}_{\mathcal{C}}^{A_2,B_2} \circ g \circ \mathcal{E}^{B_1,A_1} \circ \mathcal{E}^{A_1,B_1} \circ f \circ \mathcal{E}_{\mathcal{C}}^{B_0,A_0} = \iota_{B_1,B_2} g \circ \iota_{B_0,B_1} f$. □

DEFINITION 2100. We will turn the category $\mathcal{C}$ into a semigroup $\mathcal{UC}$ (*the semigroup of unfixed morphisms*) by the formula $[g] \circ [f] = [g \circ f]$ whenever f and g are composable morphisms.

We need to prove that $[g] \circ [f]$ does not depend on choice of f and g (provided that f and g are composable). We also need to prove that $[g] \circ [f]$ is always defined for every morphisms (not necessarily composable) f and g. That the resulting structure is a semigroup (that is, $\circ$ is associative) is then obvious.

PROOF. That $[g] \circ [f]$ is defined in at least one way for every morphisms f and g is simple to prove. Just consider the morphisms $f' = \iota_{\operatorname{Src} f, \operatorname{Dst} f \sqcup \operatorname{Src} g} f \sim f$ and $g' = \iota_{\operatorname{Dst} f \sqcup \operatorname{Src} g, \operatorname{Dst} g} g \sim g$. Then we can take $[g] \circ [f] = [g' \circ f']$.

It remains to prove that $[g] \circ [f]$ does not depend on choice of f and g. Really, take arbitrary composable pairs of morphisms $(f_0 : A_0 \to B_0, g_0 : B_0 \to C_0)$ and $(f_1 : A_1 \to B_1, g_1 : B_1 \to C_1)$ such that $f_0 \sim f_1$ and $g_0 \sim g_1$. It remains to prove that $g_0 \circ f_0 \sim g_1 \circ f_1$. We have

$$\iota_{B_0 \sqcup B_1, C_0 \sqcup C_1} g_0 \circ \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f_0 = (\text{proposition } 2099) = \mathcal{E}_{\mathcal{C}}^{C_0, C_0 \sqcup C_1} \circ g_0 \circ f_0 \circ \mathcal{E}_{\mathcal{C}}^{A_0 \sqcup A_1, B_0} = \iota_{A_0 \sqcup A_1, C_0 \sqcup C_1}(g_0 \circ f_0).$$

Similarly

$$\iota_{B_0 \sqcup B_1, C_0 \sqcup C_1} g_1 \circ \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f_1 = \iota_{A_0 \sqcup A_1, C_0 \sqcup C_1}(g_1 \circ f_1).$$

But

$$\iota_{B_0 \sqcup B_1, C_0 \sqcup C_1} g_0 \circ \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f_0 = \iota_{B_0 \sqcup B_1, C_0 \sqcup C_1} g_1 \circ \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f_1$$

thus having $\iota_{A_0 \sqcup A_1, C_0 \sqcup C_1}(g_0 \circ f_0) = \iota_{A_0 \sqcup A_1, C_0 \sqcup C_1}(g_1 \circ f_1)$ and so $g_0 \circ f_0 \sim g_1 \circ f_1$. □

6.2. Restricted identities.

DEFINITION 2101. *Restricted identity* for unfixed morphisms is defined as: $\mathrm{id}_X = [\mathrm{id}_X^{\mathcal{C}(A,B)}]$ for an $X \sqsubseteq [A] \sqcap [B]$.

We need to prove that it does not depend on the choice of A and B.

PROOF. Let $\mathfrak{A} \ni X \sqsubseteq [A_0] \sqcap [B_0]$ and $\mathfrak{A} \ni X \sqsubseteq [A_1] \sqcap [B_1]$ for $A_0, B_0, A_1, B_1 \in \mathfrak{Z}$. We need to prove $\mathrm{id}_X^{\mathcal{C}(A_0,B_0)} \sim \mathrm{id}_X^{\mathcal{C}(A_1,B_1)}$.

Really, $\iota_{A_1,B_1} \mathrm{id}_X^{\mathcal{C}(A_0,B_0)} = \mathcal{E}^{B_0,B_1} \circ \mathrm{id}_X^{\mathcal{C}(A_0,B_0)} \circ \mathcal{E}^{A_1,A_0} = \mathrm{id}_{[B_0] \sqcap [B_1]}^{\mathcal{C}(B_0,B_1)} \circ \mathrm{id}_X^{\mathcal{C}(A_0,B_0)} \circ \mathrm{id}_{[A_0] \sqcap [A_1]}^{\mathcal{C}(A_1,A_0)} = \mathrm{id}_{[A_0] \sqcap [A_1] \sqcap [B_0] \sqcap [B_1] \sqcap X}^{\mathcal{C}(A_1,B_1)} = \mathrm{id}_X^{\mathcal{C}(A_1,B_1)}$. Similarly $\iota_{A_0,B_0} \mathrm{id}_X^{\mathcal{C}(A_1,B_1)} = \mathrm{id}_X^{\mathcal{C}(A_0,B_0)}$.

So $\mathrm{id}_X^{\mathcal{C}(A_0,B_0)} \sim \mathrm{id}_X^{\mathcal{C}(A_1,B_1)}$. □

PROPOSITION 2102. $\mathrm{id}_Y \circ \mathrm{id}_X = \mathrm{id}_{X \sqcap Y}$ for every $X, Y \in \mathfrak{A}$.

Proof. Take arbitrary $\mathrm{id}_X^{\mathcal{C}(A,B_0)} \in \mathrm{id}_X$ and $\mathrm{id}_Y^{\mathcal{C}(B_1,C)} \in \mathrm{id}_Y$.

Obviously, $\mathrm{id}_X^{\mathcal{C}(A,B_0 \sqcup B_1)} \in \mathrm{id}_X$ and $\mathrm{id}_Y^{\mathcal{C}(B_0 \sqcup B_1,C)} \in \mathrm{id}_Y$. Thus $\mathrm{id}_Y \circ \mathrm{id}_X = [\mathrm{id}_Y^{\mathcal{C}(B_0 \sqcup B_1,C)}] \circ [\mathrm{id}_X^{\mathcal{C}(A,B_0 \sqcup B_1)}] = [\mathrm{id}_{X \sqcap Y}^{\mathcal{C}(A,C)}] = \mathrm{id}_{X \sqcap Y}$. □

6.3. Poset of unfixed morphisms.

Lemma 2103. $f \sqsubseteq g \Rightarrow \iota_{A,B} f \sqsubseteq \iota_{A,B} g$ for every morphisms f and g such that $\operatorname{Src} f = \operatorname{Src} g$ and $\operatorname{Dst} f = \operatorname{Dst} g$.

Proof. $\iota_{A,B} f \sqsubseteq \iota_{A,B} g \Leftrightarrow \mathcal{E}^{\operatorname{Dst} f, B} \circ f \circ \mathcal{E}^{A, \operatorname{Src} f} \sqsubseteq \mathcal{E}^{\operatorname{Dst} g, B} \circ g \circ \mathcal{E}^{A, \operatorname{Src} g} \Leftrightarrow \mathrm{id}_{[B] \sqcap [\operatorname{Dst} f]}^{\mathcal{C}(\operatorname{Dst} f, B)} \circ f \circ \mathrm{id}_{[A] \sqcap [\operatorname{Src} f]}^{\mathcal{C}(A, \operatorname{Src} f)} \sqsubseteq \mathrm{id}_{[B] \sqcap [\operatorname{Dst} g]}^{\mathcal{C}(\operatorname{Dst} g, B)} \circ g \circ \mathrm{id}_{[A] \sqcap [\operatorname{Src} g]}^{\mathcal{C}(A, \operatorname{Src} g)} \Leftarrow f \sqsubseteq g$ because $\mathrm{id}_{[B] \sqcap [\operatorname{Dst} f]}^{\mathcal{C}(\operatorname{Dst} f, B)} = \mathrm{id}_{[B] \sqcap [\operatorname{Dst} g]}^{\mathcal{C}(\operatorname{Dst} g, B)}$ and $\mathrm{id}_{[A] \sqcap [\operatorname{Src} f]}^{\mathcal{C}(A, \operatorname{Src} f)} = \mathrm{id}_{[A] \sqcap [\operatorname{Src} g]}^{\mathcal{C}(A, \operatorname{Src} g)}$. □

Corollary 2104.

1°. $f_0 \sqsubseteq g_0 \wedge f_0 \sim f_1 \wedge g_0 \sim g_1 \Rightarrow f_1 \sqsubseteq g_1$ whenever $\operatorname{Src} f_0 = \operatorname{Src} g_0$ and $\operatorname{Dst} f_0 = \operatorname{Dst} g_0$ and $\operatorname{Src} f_1 = \operatorname{Src} g_1$ and $\operatorname{Dst} f_1 = \operatorname{Dst} g_1$.

2°. $f_0 \sqsubseteq g_0 \Leftrightarrow f_1 \sqsubseteq g_1$ whenever $\operatorname{Src} f_0 = \operatorname{Src} g_0$ and $\operatorname{Dst} f_0 = \operatorname{Dst} g_0$ and $\operatorname{Src} f_1 = \operatorname{Src} g_1$ and $\operatorname{Dst} f_1 = \operatorname{Dst} g_1$ and $f_0 \sim f_1 \wedge g_0 \sim g_1$.

Proof.

1°. Because $f_1 = \iota_{\operatorname{Src} f_1, \operatorname{Dst} f_1} f_0$ and $g_1 = \iota_{\operatorname{Src} g_1, \operatorname{Dst} g_1} f_0$.

2°. A consequence of the previous.

□

The above corollary warrants validity of the following definition:

Definition 2105. The order on the set of unfixed morphisms is defined by the formula $[f] \sqsubseteq [g] \Leftrightarrow f \sqsubseteq g$ whenever $\operatorname{Src} f = \operatorname{Src} g \wedge \operatorname{Dst} f = \operatorname{Dst} g$.

It is really an order:

Proof.

Reflexivity. Obvious.

Transitivity. Obvious.

Antisymmetry. Let $[f] \sqsubseteq [g]$ and $[g] \sqsubseteq [f]$ and $\operatorname{Src} f = \operatorname{Src} g \wedge \operatorname{Dst} f = \operatorname{Dst} g$. Then $f \sqsubseteq g$ and $g \sqsubseteq f$ and thus $f = g$ so having $[f] = [g]$.

□

Obvious 2106. $f \mapsto [f]$ is an order embedding from the set $\mathcal{C}(A, B)$ to unfixed morphisms, for every objects A, B.

Proposition 2107. If S is a set of parallel morphisms of a partially ordered category with an equivalence relation respecting the order, then

1°. $\bigsqcap_{X \in S} [X]$ exists and $\bigsqcap_{X \in S} [X] = [\bigsqcap S]$;

2°. $\bigsqcup_{X \in S} [X]$ exists and $\bigsqcup_{X \in S} [X] = [\bigsqcup S]$.

Proof.

1°. $[\bigsqcap S] \sqsubseteq [X]$ for every $X \in S$ because $\bigsqcap S \sqsubseteq X$.

Let now $L \sqsubseteq [X]$ for every $X \in S$ for an equivalence class L. Then $L \sqsubseteq [\bigsqcap S]$ because $l \sqsubseteq \bigsqcap S$ for $l \in L$ because $l \sqsubseteq X$ for every $X \in S$.

Thus $[\bigsqcap S]$ is the greatest lower bound of $\left\{ \frac{[X]}{X \in S} \right\}$.

2°. By duality.

□

Proposition 2108.

1°. If every Hom-set is a join-semilattice, then the poset of unfixed morphism is a join-semilattice.
2°. If every Hom-set is a join-semilattice, then the poset of unfixed morphism is a meet-semilattice.

PROOF. Let f and g be arbitrary morphisms.

$$[f]\sqcup[g]=[\iota_{\operatorname{Src} f\sqcup\operatorname{Src} g,\operatorname{Dst} f\sqcup\operatorname{Dst} g}f]\sqcup[\iota_{\operatorname{Src} f\sqcup\operatorname{Src} g,\operatorname{Dst} f\sqcup\operatorname{Dst} g}g]= \text{(obvious 2106)}=[\iota_{\operatorname{Src} f\sqcup\operatorname{Src} g,\operatorname{Dst} f\sqcup\operatorname{Dst} g}f\sqcup\iota_{\operatorname{Src} f\sqcup\operatorname{Src} g,\operatorname{Dst} f\sqcup\operatorname{Dst} g}g]$$

and

$$[f]\sqcap[g]=[\iota_{\operatorname{Src} f\sqcup\operatorname{Src} g,\operatorname{Dst} f\sqcup\operatorname{Dst} g}f]\sqcap[\iota_{\operatorname{Src} f\sqcup\operatorname{Src} g,\operatorname{Dst} f\sqcup\operatorname{Dst} g}g]= \text{(obvious 2106)}=[\iota_{\operatorname{Src} f\sqcup\operatorname{Src} g,\operatorname{Dst} f\sqcup\operatorname{Dst} g}f\sqcap\iota_{\operatorname{Src} f\sqcup\operatorname{Src} g,\operatorname{Dst} f\sqcup\operatorname{Dst} g}g].$$

□

COROLLARY 2109. If every Hom-set is a lattice, then the poset of unfixed morphisms is a lattice.

THEOREM 2110. Meet of nonempty set of unfixed morphisms exists provided that the orders of Hom-sets are posets, every nonempty subset of which has a meet, and our category is with ordered domain and image and that morphisms $\mathcal{E}$ are metamonovalued and metainjective.

PROOF. Let S be a nonempty set of unfixed morphisms. Take an arbitrary unfixed morphism $f\in S$. Take an arbitrary $F\in f$. Let $A=\operatorname{Src} F$ and $B=\operatorname{Dst} F$. $\bigsqcap S=\bigsqcap\langle f\sqcap\rangle^* S=\bigsqcap\langle [F]\sqcap\rangle^* S=\bigsqcap\left\{\frac{[F]\sqcap[G]}{g\in S,G\in g}\right\}=\bigsqcap\left\{\frac{[\iota_{A\sqcup\operatorname{Src} G,B\sqcup\operatorname{Dst} G}F\sqcap\iota_{A\sqcup\operatorname{Src} G,B\sqcup\operatorname{Dst} G}G]}{g\in S,G\in g}\right\}$

We will prove $\iota_{A\sqcup\operatorname{Src} G,B\sqcup\operatorname{Dst} G}F\sqcap\iota_{A\sqcup\operatorname{Src} G,B\sqcup\operatorname{Dst} G}G\sim F\sqcap\iota_{A,B}G$.

$\iota_{A\sqcup\operatorname{Src} G,B\sqcup\operatorname{Dst} G}F\sqcap\iota_{A\sqcup\operatorname{Src} G,B\sqcup\operatorname{Dst} G}G\sqsubseteq\iota_{A\sqcup\operatorname{Src} G,B\sqcup\operatorname{Dst} G}F$ and $\iota_{A\sqcup\operatorname{Src} G,B\sqcup\operatorname{Dst} G}\iota_{A,B}\iota_{A\sqcup\operatorname{Src} G,B\sqcup\operatorname{Dst} G}F=\iota_{A\sqcup\operatorname{Src} G,B\sqcup\operatorname{Dst} G}F$, thus by being with ordered domain and image

$$\begin{aligned}&\iota_{A\sqcup\operatorname{Src} G,B\sqcup\operatorname{Dst} G}F\sqcap\iota_{A\sqcup\operatorname{Src} G,B\sqcup\operatorname{Dst} G}G=\\&\quad\iota_{A\sqcup\operatorname{Src} G,B\sqcup\operatorname{Dst} G}\iota_{A,B}(\iota_{A\sqcup\operatorname{Src} G,B\sqcup\operatorname{Dst} G}F\sqcap\iota_{A\sqcup\operatorname{Src} G,B\sqcup\operatorname{Dst} G}G)=\\&\quad\text{(by being metamonovalued and metainjective)}=\\&\quad\iota_{A\sqcup\operatorname{Src} G,B\sqcup\operatorname{Dst} G}(\iota_{A,B}\iota_{A\sqcup\operatorname{Src} G,B\sqcup\operatorname{Dst} G}F\sqcap\iota_{A,B}\iota_{A\sqcup\operatorname{Src} G,B\sqcup\operatorname{Dst} G}G)=\\&\quad\iota_{A\sqcup\operatorname{Src} G,B\sqcup\operatorname{Dst} G}(\iota_{A,B}F\sqcap\iota_{A,B}G)\sim\iota_{A,B}F\sqcap\iota_{A,B}G=F\sqcap\iota_{A,B}G.\end{aligned}$$

Due the proved equivalence we have $\bigsqcap S=\bigsqcap\left\{\frac{[F\sqcap\iota_{A,B}G]}{g\in S,G\in g}\right\}$. Now we can apply proposition 2107: $\bigsqcap S=\left[\bigsqcap\left\{\frac{F\sqcap\iota_{A,B}G}{g\in S,G\in g}\right\}\right]$. We have provided an explicit formula for $\bigsqcap S$. □

The poset of unfixed morphisms may be not a complete lattice even if every Hom-set is a complete lattice. We will show this below for funcoids.

6.4. Domain and image of unfixed morphisms.

PROPOSITION 2111. $\operatorname{IM} f=\left\{\frac{Y\in\mathfrak{Z}}{\mathrm{id}_Y\circ[f]=[f]}\right\}$; $\operatorname{DOM} f=\left\{\frac{X\in\mathfrak{Z}}{[f]\circ\mathrm{id}_X=[f]}\right\}$.

PROOF. We will prove only the first, as the second is similar. $\mathrm{id}_Y\circ[f]=[f]\Leftrightarrow\mathrm{id}_Y^{\mathcal{C}(Y\sqcup\operatorname{Dst} f,Y\sqcup\operatorname{Dst} f)}\circ\mathcal{E}^{\operatorname{Dst} f,Y\sqcup\operatorname{Dst} f}\circ f=\mathcal{E}^{\operatorname{Dst} f,Y\sqcup\operatorname{Dst} f}\circ f\Leftrightarrow\mathrm{id}_{[Y]\sqcap[\operatorname{Dst} f]}^{\mathcal{C}(\operatorname{Dst} f,Y\sqcup\operatorname{Dst} f)}\circ f=\mathcal{E}^{\operatorname{Dst} f,Y\sqcup\operatorname{Dst} f}\circ f\Leftrightarrow\mathcal{E}^{Y\sqcup\operatorname{Dst} f,\operatorname{Dst} f}\circ\mathrm{id}_{[Y]\sqcap[\operatorname{Dst} f]}^{\mathcal{C}(\operatorname{Dst} f,Y\sqcup\operatorname{Dst} f)}\circ f=f\Leftrightarrow\mathrm{id}_{[Y]\sqcap[\operatorname{Dst} f]}^{\mathcal{C}(\operatorname{Dst} f,\operatorname{Dst} f)}\circ f=f Y\in\operatorname{IM} f$. □

The above proposition allows to define:

DEFINITION 2112. $\operatorname{DOM} f = \operatorname{DOM} F$ and $\operatorname{IM} f = \operatorname{IM} F$ for $F \in f$.

6.5. Rectangular restriction.

PROPOSITION 2113. $\iota_{A,B} f = \iota_{A,B} g$ if $f \sim g$.

PROOF. Let $f \sim g$. Then $g = \iota_{\operatorname{Src} g,\operatorname{Dst} g} f$. So $\iota_{A,B} g = \iota_{A,B} \iota_{\operatorname{Src} g,\operatorname{Dst} g} f \sqsubseteq$ (proposition 2090) $\sqsubseteq \iota_{A,B} f$. Similarly, $\iota_{A,B} f \sqsubseteq \iota_{A,B} g$. So $\iota_{A,B} f = \iota_{A,B} g$. □

The above proposition allows to define:

DEFINITION 2114. $\iota_{A,B} F = \iota_{A,B} f$ for an unfixed morphism F and arbitrary $f \in F$.

DEFINITION 2115. $F\square_{A,B} = [\iota_{A,B} F]$ for every unfixed morphism F.

PROPOSITION 2116. $F\square_{A,B} = \operatorname{id}_B \circ F \circ \operatorname{id}_A$ for every unfixed morphism F and objects A and B.

PROOF. Take $f \in F$. $F\square_{A,B} = [\iota_{A,B} F] = [\iota_{A,B} f] = [\mathcal{E}^{\operatorname{Dst} f,B} \circ f \circ \mathcal{E}^{A,\operatorname{Src} f}] = [\operatorname{id}^{\mathcal{C}(\operatorname{Dst} f,B)}_{B\sqcap\operatorname{Dst} f} \circ f \circ \operatorname{id}^{\mathcal{C}(A,\operatorname{Src} f)}_{A\sqcap\operatorname{Src} f}] = [\operatorname{id}^{\mathcal{C}(\operatorname{Dst} f,B)}_{B} \circ \operatorname{id}^{\mathcal{C}(\operatorname{Dst} f,\operatorname{Dst} f)}_{\operatorname{Dst} f} \circ f \circ \operatorname{id}^{\mathcal{C}(\operatorname{Src} f,\operatorname{Src} f)}_{\operatorname{Src} f} \circ \operatorname{id}^{\mathcal{C}(A,\operatorname{Src} f)}_{A}] = [\operatorname{id}^{\mathcal{C}(\operatorname{Dst} f,B)}_{B} \circ f \circ \operatorname{id}^{\mathcal{C}(A,\operatorname{Src} f)}_{A}] = [\operatorname{id}^{\mathcal{C}(\operatorname{Dst} f,B)}_{B}] \circ [f] \circ [\operatorname{id}^{\mathcal{C}(A,\operatorname{Src} f)}_{A}] = \operatorname{id}_B \circ F \circ \operatorname{id}_A$. □

PROPOSITION 2117. $f\square_{A_0,B_0}\square_{A_1,B_1} = f\square_{A_0\sqcap A_1,A_1\sqcap B_1}$.

PROOF. From the previous $f\square_{A_0,B_0}\square_{A_1,B_1} = \operatorname{id}_{B_1} \circ \operatorname{id}_{B_0} \circ f \circ \operatorname{id}_{A_0} \circ \operatorname{id}_{A_1} = \operatorname{id}_{B_0\sqcap B_1} \circ f \circ \operatorname{id}_{A_0\sqcap A_1} = f\square_{A_0\sqcap A_1,A_1\sqcap B_1}$. □

DEFINITION 2118. $f|_X = f \circ \operatorname{id}_X$ for every unfixed morphism f and $X \in \mathfrak{A}$.

OBVIOUS 2119. $(f|_X)|_Y = f_{X\sqcap Y}$.

6.6. Algebraic properties of the lattice of unfixed morphisms. The following proposition allows to easily prove algebraic properties (cf. distributivity) of the poset of unfixed morphisms:

THEOREM 2120. The following are mutually inverse bijections:

1°. Let A and B be objects. $f \mapsto [f]$ and $F \mapsto \iota_{A,B} F$ are mutually inverse order isomorphisms between $\left\{ \frac{f\in\text{unfixed morphisms}}{A\in\operatorname{DOM} f, B\in\operatorname{IM} f} \right\}$ and $\mathcal{C}(A,B)$. If $A = B$ they are also semigroup isomorphisms.

2°. Let T be an unfixed morphism. $f \mapsto [f]$ and $F \mapsto \iota_{\operatorname{Src} t,\operatorname{Dst} t} F$ are mutually inverse order isomorphisms between the lattice DT and Dt whenever $t \in T$.

PROOF. We will prove that these functions are mutually inverse bijections. That they are order-preserving is obvious.

1°. $\iota_{A,B} F \in \mathcal{C}(A,B)$ is obvious.

We need to prove that $[f] \in \left\{ \frac{f\in\text{unfixed morphisms}}{A\in\operatorname{DOM} f, B\in\operatorname{IM} f} \right\}$. For this it's enough to prove $A \in \operatorname{DOM}[f] \wedge B \in \operatorname{IM}[f]$ what is the same as $A \in \operatorname{DOM} f \wedge B \in \operatorname{IM} f$ what follows from proposition 2071.

Because $f \mapsto [f]$ is an injection, it is enough[1] to prove that $\iota_{A,B}[f] = f$. Really, $\iota_{A,B}[f] = \iota_{A,B} f = f$.

That they are semigroup isomorphisms follows from the already proved formula $[g \circ f] = [g] \circ [f]$.

[1] https://math.stackexchange.com/a/3007051/4876

2°. Because of the previous, it is enough to prove that $[f] \in DT \Leftrightarrow f \in Dt$. Really, it is equivalent to $[f] \sqsubseteq T \Leftrightarrow f \sqsubseteq t$ what is obvious.

□

PROPOSITION 2121. If every Hom-set is a distributive lattice, then the poset of unfixed morphisms is a distributive lattice.

PROOF. It follows from the above isomorphism. □

PROPOSITION 2122. If every Hom-set is a co-brouwerian lattice, then the poset of unfixed morphisms is a co-brouwerian lattice.

PROOF. It follows from the above isomorphism and the definition of pseudodifference. □

PROPOSITION 2123. If every Hom-set is a lattice with quasidifference, then the poset of unfixed morphisms is a lattice with quasidifference.

PROOF. It follows from the above isomorphism and the definition of quasidifference. □

PROPOSITION 2124.

1°. If every Hom-set is an atomic lattice, then the poset of unfixed morphisms is an atomic lattice.
2°. If every Hom-set is an atomistic lattice, then the poset of unfixed morphisms is an atomistic lattice.

PROOF. Follows from the above isomorphism. □

6.7. Binary product morphism.

DEFINITION 2125. For a category $\mathcal{C}$ with binary product morphism and $X, Y \in \mathfrak{A}$ define $X \times Y = [X \times_{A,B} Y]$ where $A \in \mathfrak{Z}$, $[A] \sqsupseteq X$, $B \in \mathfrak{Z}$, $[B] \sqsupseteq Y$. (Such A and B exist by an axiom of categories with restricted identities.)

We need to prove validity of this definition:

PROOF. Let $A_0 \in \mathfrak{Z}$, $[A_0] \sqsupseteq X$, $B_0 \in \mathfrak{Z}$, $[B_0] \sqsupseteq Y$, $A_1 \in \mathfrak{Z}$, $[A_1] \sqsupseteq X$, $B_1 \in \mathfrak{Z}$, $[B_1] \sqsupseteq Y$. We need to prove $X \times_{A_0,B_0} Y \sim X \times_{A_1,B_1} Y$, but it trivially follows from an axiom in the definition of category with binary product morphism. □

PROPOSITION 2126. $(X_0 \times Y_0) \sqcap (X_1 \times Y_1) = (X_0 \sqcap X_1) \times (Y_0 \sqcap Y_1)$ for every $X_0, X_1, Y_0, Y_1 \in \mathfrak{A}$.

PROOF. Take $A_0 \in \mathfrak{Z}$, $[A_0] \sqsupseteq X_0$, $B_0 \in \mathfrak{Z}$, $[B_0] \sqsupseteq Y_0$, $A_1 \in \mathfrak{Z}$, $[A_1] \sqsupseteq X_1$, $B_1 \in \mathfrak{Z}$, $[B_1] \sqsupseteq Y_1$.

Then $(X_0 \times Y_0) \sqcap (X_1 \times Y_1) = [X_0 \times_{A_0 \sqcup A_1, B_0 \sqcup B_1} Y_0] \sqcap [X_1 \times_{A_0 \sqcup A_1, B_0 \sqcup B_1} Y_1] = [(X_0 \times_{A_0 \sqcup A_1, B_0 \sqcup B_1} Y_0) \sqcap (X_1 \times_{A_0 \sqcup A_1, B_0 \sqcup B_1} Y_1)] = [(X_0 \sqcap X_1) \times_{A_0 \sqcup A_1, B_0 \sqcup B_1} (Y_0 \sqcap Y_1)] = (X_0 \sqcap X_1) \times (Y_0 \sqcap Y_1)$. □

PROPOSITION 2127. $f \square_{A,B} = f \sqcap (A \times B)$.

PROOF. Take $F \in f$. Let $F' = \iota_{A \sqcup \operatorname{Src} F, B \sqcup \operatorname{Dst} F} F$. We have $F' \in f$. $f \square_{A,B} = [\iota_{A,B} F'] = [\mathcal{E}^{B \sqcup \operatorname{Dst} F, B} \circ F' \circ \mathcal{E}^{A, A \sqcup \operatorname{Src} F}] = [\mathrm{id}^{\mathcal{C}(B \sqcup \operatorname{Dst} F, B)}_{[B]} \circ F' \circ \mathrm{id}^{\mathcal{C}(A, A \sqcup \operatorname{Src} F)}_{[A]}] = [\mathrm{id}^{\mathcal{C}(B \sqcup \operatorname{Dst} F, B)}_{[B]}] \circ [F'] \circ [\mathrm{id}^{\mathcal{C}(A, A \sqcup \operatorname{Src} F)}_{[A]}] = [\mathrm{id}^{\mathcal{C}(B \sqcup \operatorname{Dst} F, B \sqcup \operatorname{Dst} F)}_{[B]}] \circ [F'] \circ [\mathrm{id}^{\mathcal{C}(A \sqcup \operatorname{Src} F, A \sqcup \operatorname{Src} F)}_{[A]}] = [\mathrm{id}^{\mathcal{C}(B \sqcup \operatorname{Dst} F, B \sqcup \operatorname{Dst} F)}_{[B]} \circ F' \circ \mathrm{id}^{\mathcal{C}(A \sqcup \operatorname{Src} F, A \sqcup \operatorname{Src} F)}_{[A]}] = [F' \sqcap (A \times_{A \sqcup \operatorname{Src} F, B \sqcup \operatorname{Dst} F} B)] = [F'] \sqcap [A \times_{A \sqcup \operatorname{Src} F, B \sqcup \operatorname{Dst} F} B] = f \sqcap (A \times B)$. □

7. Examples of categories with restricted identities

7.1. Category Rel. Category **Rel** of relations between small sets can be considered as a category with restricted identities with $\mathfrak{Z} = \mathfrak{A}$ being the set of all small sets, projection being the identity function and restricted identity being the identity relation between the given sets.

Moreover it is a category with binary product morphism with usual Cartesian product (??prove).

Proofs of this are trivial.

7.2. Category FCD. FiXme: It is FCD, not $\mathcal{C}$.

Category FCD can be considered as a category with restricted identities with $\mathfrak{Z}$ being the set of all small sets, $\mathfrak{A}$ is the set of unfixed filters, projection being the projection function for the equivalence classes of filters, restricted identity being defined by the formulas

$$\left\langle \mathrm{id}_{\mathcal{F}}^{\mathcal{C}(A,B)} \right\rangle \mathcal{X} = ([\mathcal{X}] \sqcap \mathcal{F}) \div B; \quad \left\langle (\mathrm{id}_{\mathcal{F}}^{\mathcal{C}(A,B)})^{-1} \right\rangle \mathcal{Y} = ([\mathcal{Y}] \sqcap \mathcal{F}) \div A$$

(whenever $\mathcal{F} \sqsubseteq [A] \sqcap [B]$).

We need to prove that this really defines a funcoid.

Proof. $\mathcal{Y} \not\asymp \left\langle \mathrm{id}_{\mathcal{F}}^{\mathcal{C}(A,B)} \right\rangle \mathcal{X} \Leftrightarrow \mathcal{Y} \not\asymp ([\mathcal{X}] \sqcap \mathcal{F}) \div B \Leftrightarrow \mathcal{Y} \not\asymp (\mathcal{X} \div B) \sqcap (\mathcal{F} \div B) \Leftrightarrow [\mathcal{Y}] \not\asymp [\mathcal{X}] \sqcap \mathcal{F}$. Similarly $\left\langle (\mathrm{id}_{\mathcal{F}}^{\mathcal{C}(A,B)})^{-1} \right\rangle \mathcal{Y} \Leftrightarrow [\mathcal{X}] \not\asymp [\mathcal{Y}] \sqcap \mathcal{F}$.

Thus $\mathcal{Y} \not\asymp \left\langle \mathrm{id}_{\mathcal{F}}^{\mathcal{C}(A,B)} \right\rangle \mathcal{X} \Leftrightarrow \mathcal{X} \not\asymp \left\langle (\mathrm{id}_{\mathcal{F}}^{\mathcal{C}(A,B)})^{-1} \right\rangle \mathcal{Y}$. □

We need to prove that the restricted identities conform to the axioms:

Proof. The first five axioms are obvious. Let's prove the remaining ones:

$\mathrm{id}_{[A]}^{\mathcal{C}(A,A)} = 1_A^{\mathcal{C}}$ because $\left\langle \mathrm{id}_{[A]}^{\mathcal{C}(A,A)} \right\rangle \mathcal{X} = ([\mathcal{X}] \sqcap [A]) \div A = [\mathcal{X}] \div A = \mathcal{X}$.

$\mathrm{id}_Y^{\mathcal{C}(B,C)} \circ \mathrm{id}_X^{\mathcal{C}(A,B)} = \mathrm{id}_{X \sqcap Y}^{\mathcal{C}(A,C)}$ because $\left\langle \mathrm{id}_Y^{\mathcal{C}(B,C)} \circ \mathrm{id}_X^{\mathcal{C}(A,B)} \right\rangle \mathcal{X} = \left\langle \mathrm{id}_Y^{\mathcal{C}(B,C)} \right\rangle \left\langle \mathrm{id}_X^{\mathcal{C}(A,B)} \right\rangle \mathcal{X} = \left\langle \mathrm{id}_Y^{\mathcal{C}(B,C)} \right\rangle (([\mathcal{X}] \sqcap X) \div B) = ([([\mathcal{X}] \sqcap X) \div B] \sqcap Y) \div C = (([\mathcal{X}] \sqcap X \sqcap Y) \div B) \div C =$ (because $[\mathcal{X}] \sqcap X \sqcap Y \sqsubseteq [B]$) $= ([\mathcal{X}] \sqcap X \sqcap Y) \div C = \left\langle \mathrm{id}_{X \sqcap Y}^{\mathcal{C}(A,C)} \right\rangle \mathcal{X}$.

$\forall A \in \mathfrak{A} \exists B \in \mathfrak{Z} : A \sqsubseteq [B]$ is obvious. □

Proposition 2128. $\mathcal{E}_{\mathsf{FCD}}^{A,B} = (A, B, \lambda \mathcal{X} \in \mathfrak{F}(A) : \mathcal{X} \div B, \lambda \mathcal{Y} \in \mathfrak{F}(B) : \mathcal{Y} \div A)$ for objects $A \subseteq B$ of FCD.

Proof. Take $\mathcal{F} = [A] \sqcap [B]$. Then $\mathcal{F} \sqsupseteq [\mathcal{X}]$ and $\mathcal{F} \sqsupseteq [\mathcal{Y}]$, thus $[\mathcal{X}] \sqcap \mathcal{F} = [\mathcal{X}]$ and $[\mathcal{Y}] \sqcap \mathcal{F} = [\mathcal{Y}]$. So, it follows from the above. □

Proposition 2129. $\mathrm{id}_X^{\mathsf{FCD}(A,A)} = \mathrm{id}_{X \div A}^{\mathsf{FCD}}$ whenever $A \in \mathfrak{Z}$ and $\mathfrak{A} \ni X \sqsubseteq [A]$.

Proof. $\left\langle \mathrm{id}_X^{\mathsf{FCD}(A,A)} \right\rangle \mathcal{X} = ([\mathcal{X}] \sqcap X) \div A = ([\mathcal{X}] \div A) \sqcap (X \div A) = \mathcal{X} \sqcap (X \div A) = \left\langle \mathrm{id}_{X \div A}^{\mathsf{FCD}} \right\rangle \mathcal{X}$ (used bijections for unfixed filters) for every $\mathcal{X} \in \mathscr{F}(A)$. □

Definition 2130. Category FCD can be considered as a category with binary product morphism with the binary product defined as: $\mathcal{X} \times_{A,B} \mathcal{Y} = (\mathcal{X} \div A) \times^{\mathsf{FCD}} (\mathcal{Y} \div B)$ for every unfixed filters $\mathcal{X}$ and $\mathcal{Y}$.

It is really a binary product morphism:

Proof. Need to prove the axioms:

1°. $f \sqcap (X \times_{A,B} Y) = f \sqcap ((X \div A) \times^{\mathsf{FCD}} (Y \div B)) = \mathrm{id}^{\mathsf{FCD}}_{Y \div B} \circ f \circ \mathrm{id}^{\mathsf{FCD}}_{X \div A} = \mathrm{id}^{\mathsf{FCD}(B,B)}_{Y} \circ f \circ \mathrm{id}^{\mathsf{FCD}(A,A)}_{X}$.

2°. Let unfixed filters $X \sqsubseteq [A_0] \sqcap [A_1]$ and $Y \sqsubseteq [B_0] \sqcap [B_1]$. Then for $\mathcal{X} \in \mathscr{F}(A_1)$ we have $\langle \iota_{A_1,B_1}(X \times_{A_0,B_0} Y)\rangle\mathcal{X} = \langle \mathcal{E}^{\mathcal{C}(B_0,B_1)}\rangle\langle X \times_{A_0,B_0} Y\rangle\langle \mathcal{E}^{\mathcal{C}(A_1,A_0)}\rangle\mathcal{X} = (\langle X \times_{A_0,B_0} Y\rangle(\mathcal{X} \div A_0)) \div B_1 = (\langle (X \div A_0) \times^{\mathsf{FCD}} (Y \div B_0)\rangle(\mathcal{X} \div A_0)) \div B_1$.

On the other hand, $\langle X \times_{A_1,B_1} Y\rangle\mathcal{X} = \langle (X \div A_1) \times^{\mathsf{FCD}} (Y \div B_1)\rangle\mathcal{X}$

If $[\mathcal{X}] \asymp X$ then (use isomorphisms) $\mathcal{X} \asymp X \div A_1$ and $\mathcal{X} \div A_0 \asymp X \div A_0$. So $\langle \iota_{A_1,B_1}(X \times_{A_0,B_0} Y)\rangle\mathcal{X} = \bot$ and $\langle X \times_{A_1,B_1} Y\rangle\mathcal{X} = \bot$.

If $[\mathcal{X}] \not\asymp X$ then (use isomorphisms) $\mathcal{X} \not\asymp X \div A_1$ and $\mathcal{X} \div A_0 \not\asymp X \div A_0$. So $\langle \iota_{A_1,B_1}(X \times_{A_0,B_0} Y)\rangle\mathcal{X} = (Y \div B_0) \div B_1 = Y \div B_1$ and $\langle X \times_{A_1,B_1} Y\rangle\mathcal{X} = Y \div B_1$.

So in all cases, $\langle \iota_{A_1,B_1}(X \times_{A_0,B_0} Y)\rangle\mathcal{X} = \langle X \times_{A_1,B_1} Y\rangle\mathcal{X}$.

□

LEMMA 2131. FiXme: This lemma seems not used. $\mathcal{X} \div A = (\mathcal{X} \sqcap [A]) \div A$ for every unfixed filter $\mathcal{X}$ and small set A.

PROOF. $(\mathcal{X} \sqcap [A]) \div A = (\mathcal{X} \div A) \sqcap ([A] \div A) = ?? = (\mathcal{X} \div A) \sqcap \top^{\mathfrak{F}(A)} = \mathcal{X} \div A$. □

COROLLARY 2132. There is a pointfree funcoid p such that $\langle p\rangle\mathcal{X} = \mathcal{X} \div A$.

PROOF. Let q be the order embedding (see the diargram) from unfixed filters $\mathcal{F}$ such that $A \in \mathcal{F}$ to filters on A.

Then $\langle \mathcal{X} \div A\rangle\mathcal{X} = \langle (\mathcal{X} \sqcap [A]) \div A\rangle\mathcal{X} = \left\langle q \circ \mathrm{id}^{\mathsf{pFCD}(\text{unfixed filters})}_{[A]}\right\rangle\mathcal{X}$. □

Let f be a funcoid. Define pointfree funcoid $\mathscr{S}f$ between unfixed filters as:

DEFINITION 2133. For every unfixed filters $\mathcal{X}$ and $\mathcal{Y}$

$$\langle \mathscr{S}f\rangle\mathcal{X} = [\langle f\rangle(\mathcal{X} \div \operatorname{Src} f)]; \quad \left\langle (\mathscr{S}f)^{-1}\right\rangle\mathcal{Y} = [\left\langle f^{-1}\right\rangle(\mathcal{Y} \div \operatorname{Dst} f)].$$

It is really a pointfree funcoid:

PROOF. For an unfixed filter $\mathcal{Y}$ we have $\mathcal{Y} \not\asymp \langle \mathscr{S}f\rangle\mathcal{X} \Leftrightarrow \mathcal{Y} \not\asymp [\langle f\rangle(\mathcal{X} \div \operatorname{Src} f)] \Leftrightarrow \mathcal{Y} \div \operatorname{Dst} f \not\asymp \langle f\rangle(\mathcal{X} \div \operatorname{Src} f) \Leftrightarrow \mathcal{X} \div \operatorname{Src} f \not\asymp \left\langle f^{-1}\right\rangle(\mathcal{Y} \div \operatorname{Dst} f) \Leftrightarrow \mathcal{X} \not\asymp [\left\langle f^{-1}\right\rangle(\mathcal{Y} \div \operatorname{Dst} f)] \Leftrightarrow \mathcal{X} \not\asymp \left\langle (\mathscr{S}f)^{-1}\right\rangle\mathcal{Y}$. □

DEFINITION 2134. $\mathscr{S}F = \mathscr{S}f$ for an unfixed funcoid F and $f \in F$.

We need to prove validity of the above definition:

PROOF. Let $f, g \in F$, let $f : A_0 \to B_0$, $g : A_1 \to B_1$. Need to prove $\mathscr{S}f = \mathscr{S}g$. We have

$$\iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f = \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} g.$$

$\langle \mathscr{S}\iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f\rangle\mathcal{X} = [\langle \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f\rangle(\mathcal{X} \div (A_0 \sqcup A_1))] = [\langle \mathcal{E}^{\mathsf{FCD}(B_0, B_0 \sqcup B_1)}\rangle\langle f\rangle\langle \mathcal{E}^{\mathsf{FCD}(A_0 \sqcup A_1, A_0}\rangle(\mathcal{X} \div (A_0 \sqcup A_1))] = [(\langle f\rangle((\mathcal{X} \div (A_0 \sqcup A_1)) \div A_0)) \div B_0] = [\langle f\rangle(\mathcal{X} \div A_0)] = \langle \mathscr{S}f\rangle\mathcal{X}$.

Similarly $\langle \mathscr{S}\iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} g\rangle\mathcal{X} = \langle \mathscr{S}g\rangle\mathcal{X}$.

So $\langle \mathscr{S}f\rangle\mathcal{X} = \langle \mathscr{S}g\rangle\mathcal{X}$. □

PROPOSITION 2135.

1°. $\mathscr{S}$ from a Hom-set $\mathsf{FCD}(A, B)$ is an order embedding.
2°. $\mathscr{S}$ from the category FCD is a prefunctor.
3°. $\mathscr{S}$ from unfixed funcoids is an order embedding and a prefunctor (= semigroup homomorphism).

PROOF.

1°. $(\langle \mathscr{S} f\rangle \mathcal{X}) \div \operatorname{Dst} f = \langle f\rangle \mathcal{X}$. Thus for different f we have different $\mathcal{X} \mapsto \langle \mathscr{S} f\rangle \mathcal{X}$. So it is an injection. That it is a monotone function is obvious.

2°. $\langle \mathscr{S} g \circ \mathscr{S} f\rangle \mathcal{X} = \langle \mathscr{S} g\rangle \langle \mathscr{S} f\rangle \mathcal{X} = \langle \mathscr{S} g\rangle [\langle f\rangle (\mathcal{X} \div \operatorname{Src} f)] = [\langle g\rangle ([\langle f\rangle (\mathcal{X} \div \operatorname{Src} f)] \div \operatorname{Src} g)] = [\langle g\rangle (\langle f\rangle (\mathcal{X} \div \operatorname{Src} f) \div \operatorname{Src} g)] = [\langle g\rangle \langle f\rangle (\mathcal{X} \div \operatorname{Src} f)] = [\langle g \circ f\rangle (\mathcal{X} \div \operatorname{Src} f)] = \langle \mathscr{S} (g \circ f)\rangle \mathcal{X}$ for every composable funcoids f and g and an unfixed filter $\mathcal{X}$. Thus $\mathscr{S} g \circ \mathscr{S} f = \mathscr{S} (g \circ f)$.

3°. To prove that it is an order embedding, it is enough to show that $f \not\asymp g$ implies $\mathscr{S} f \neq \mathscr{S} g$ (monotonicity is obvious). Let $f \not\asymp g$ that is $\iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f \neq \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} g$. Then there exist filter $\mathcal{X} \in \mathfrak{F}(A_0 \sqcup A_1)$ such that $\langle \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f\rangle \mathcal{X} \neq \langle \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} g\rangle \mathcal{X}$.

Consequently, $\langle \mathscr{S} f\rangle \mathcal{X} = \langle \mathscr{S} \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f\rangle \mathcal{X} \neq \langle \mathscr{S} \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} g\rangle \mathcal{X} = \langle \mathscr{S} g\rangle \mathcal{X}$.

It remains to prove that $\mathscr{S} G \circ \mathscr{S} F = \mathscr{S} (G \circ F)$ but it is equivalent to $\mathscr{S} g \circ \mathscr{S} f = \mathscr{S} (g \circ f)$ for arbitrarily taken $f \in F$ and $g \in G$, what is already proved above.

□

LEMMA 2136. For every meet-semilattice $a \not\asymp b$ and $c \sqsupseteq b$ implies $a \sqcap c \not\asymp b$.

PROOF. Suppose $a \not\asymp b$. Then there is a non-least x such that $x \sqsubseteq a, b$. Thus $x \sqsubseteq c$, so $x \sqsubseteq a \sqcap c$. We have $a \sqcap c \not\asymp b$. □

FiXme: Since here also for reloids.

PROPOSITION 2137. $\mathscr{S} (X \times Y) = X \times^{\mathsf{pFCD}(\mathfrak{F}(\mho))} Y$ for every unfixed filters X and Y.

PROOF. $\mathscr{S} (X \times Y) = \mathscr{S} (X \times_{A,B} Y)$ for arbitrary filters A, and B such that $X \sqsubseteq [A]$ and $Y \sqsubseteq [B]$. So for every unfixed filter $\mathcal{X}$ we have $\langle \mathscr{S} (X \times Y)\rangle \mathcal{X} = \langle \mathscr{S} (X \times_{A,B} Y)\rangle \mathcal{X} = [\langle X \times_{A,B} Y\rangle (\mathcal{X} \div A)] = [\langle (X \div A) \times^{\mathsf{FCD}} (Y \div B)\rangle (\mathcal{X} \div A)]$

Thus if $\mathcal{P} \not\asymp X$ then (by the lemma) $\mathcal{P} \sqcap A \not\asymp X$; $\mathcal{P} \div A \not\asymp X \div A$; $\langle \mathscr{S} (X \times Y)\rangle \mathcal{X} = [Y \div B] = Y$.

if $\mathcal{P} \asymp X$ then $\mathcal{P} \sqcap A \asymp X$; $\mathcal{P} \div A \asymp X \div A$; $\langle \mathscr{S} (X \times Y)\rangle \mathcal{X} = [\bot] = \bot$.

So $\mathscr{S} (X \times Y) = X \times^{\mathsf{pFCD}(\mathfrak{F}(\mho))} Y$. □

PROPOSITION 2138. $\mathscr{S} \operatorname{id}_X = \operatorname{id}_X^{\mathsf{pFCD}(\mathfrak{F}(\mho))}$ for every unfixed filter X.

PROOF. For every unfixed filter $\mathcal{X}$ we for arbitrary filters A and B such that $X \sqsubseteq [A] \sqcap [B]$ have $\langle \mathscr{S} \operatorname{id}_X\rangle \mathcal{X} = \left\langle \mathscr{S} [\operatorname{id}_X^{\mathcal{C}(A,B)}]\right\rangle \mathcal{X} = \left\langle \mathscr{S} \operatorname{id}_X^{\mathcal{C}(A,B)}\right\rangle \mathcal{X} = \left[\left\langle \operatorname{id}_X^{\mathcal{C}(A,B)}\right\rangle (\mathcal{X} \div A)\right] = [([\mathcal{X} \div A] \sqcap X) \div B] = [(\mathcal{X} \sqcap X) \div B] = \mathcal{X} \sqcap X$.

Thus $\mathscr{S} \operatorname{id}_X = \operatorname{id}_X^{\mathsf{pFCD}(\mathfrak{F}(\mho))}$. □

7.3. Category RLD.

DEFINITION 2139. $f \div D = (A, B, (\operatorname{GR} f) \div D)$ for every reloid f and a binary relation D.

Category RLD can be considered as a category with restricted identities with $\mathfrak{Z}$ being the set of all small sets, $\mathfrak{A}$ is the set of unfixed filters, projection being the projection function for the equivalence classes of filters, restricted identity being defined by the formula

$$\operatorname{id}_{\mathcal{F}}^{\mathsf{RLD}(A,B)} = \operatorname{id}_{\mathcal{F} \div (A \cap B)}^{\mathsf{RLD}} \div (A \times B).$$

We need to prove that the restricted identities conform to the axioms:

PROOF. The first five axioms are obvious. Let's prove the remaining ones:
$\operatorname{id}_{[A]}^{\mathsf{RLD}(A,A)} = \operatorname{id}_{[A] \div A}^{\mathsf{RLD}} \div (A \times A) = \operatorname{id}_A^{\mathsf{RLD}} \div (A \times A) = 1_A^{\mathsf{RLD}}$.

$\mathrm{id}_Y^{\mathcal{C}(B,C)} \circ \mathrm{id}_X^{\mathcal{C}(A,B)} = \bigsqcap_{x\in\mathrm{up}\,X, y\in\mathrm{up}\,Y}(\mathrm{id}_y^{\mathcal{C}(B,C)} \circ \mathrm{id}_x^{\mathcal{C}(A,B)}) = \bigsqcap_{x\in\mathrm{up}\,X, y\in\mathrm{up}\,Y} \mathrm{id}_{x\sqcap y}^{\mathcal{C}(A,B)} = \mathrm{id}_{X\sqcap Y}^{\mathcal{C}(A,B)}$.

$\forall A\in\mathfrak{A}\exists B\in\mathfrak{Z}: A\sqsubseteq[B]$ is obvious. □

OBVIOUS 2140. $\mathcal{E}_{\mathsf{RLD}}^{A,B} = \uparrow^{\mathsf{RLD}(A,B)} \mathrm{id}_{A\cap B}$.

PROPOSITION 2141. RLD with $\mathcal{X}\times_{A,B}\mathcal{Y} = (\mathcal{X}\div A)\times^{\mathsf{RLD}}(\mathcal{X}\div B)$ for every unfixed filters $\mathcal{X}$ and $\mathcal{Y}$ is a category with binary product morphism.

PROOF. $\mathrm{id}_{\mathcal{Y}}^{\mathcal{C}(B,B)}\circ f\circ \mathrm{id}_{\mathcal{X}}^{\mathcal{C}(A,A)} = f\sqcap(\mathcal{X}\times_{A,B}\mathcal{Y})$ because $\mathrm{id}_{\mathcal{Y}}^{\mathcal{C}(B,B)}\circ f\circ \mathrm{id}_{\mathcal{X}}^{\mathcal{C}(A,A)} = (\mathrm{id}^{\mathsf{RLD}}_{\mathcal{Y}\div B}\div(B\times B))\circ f\circ(\mathrm{id}^{\mathsf{RLD}}_{\mathcal{X}\div A}\div(A\times A)) = \mathrm{id}^{\mathsf{RLD}}_{\mathcal{Y}\div B}\circ f\circ \mathrm{id}^{\mathsf{RLD}}_{\mathcal{X}\div A} = f\sqcap((\mathcal{X}\div A)\times^{\mathsf{RLD}}(\mathcal{Y}\div B)) = f\sqcap(\mathcal{X}\times_{A,B}\mathcal{Y})$.

$\iota_{A_1,B_1}(\mathcal{X}\times_{A_0,B_0}\mathcal{Y}) = \mathcal{E}^{B_0,B_1}\circ(\mathcal{X}\times_{A_0,B_0}\mathcal{Y})\circ\mathcal{E}^{A_1,A_0} = \uparrow^{\mathsf{RLD}(B_0,B_1)}\mathrm{id}_{B_0\cap B_1}((\mathcal{X}\div B_0)\times(\mathcal{Y}\div A_0))\circ\uparrow^{\mathsf{RLD}(A_1,A_0)}\mathrm{id}_{A_0\cap A_1} = ((\mathcal{X}\div B_0)\div B_1)\times((\mathcal{Y}\div A_0)\div A_1) = (\mathcal{X}\div B_1)\times(\mathcal{Y}\div A_1) = \mathcal{X}\times_{A_1,B_1}\mathcal{Y}$. □

PROPOSITION 2142. $\iota_{A,B}f = f\div(A\times B)$ for every reloid f.

PROOF. $\iota_{A,B}f = \mathcal{E}_{\mathsf{RLD}}^{\mathrm{Dst}\,f,B}\circ f\circ \mathcal{E}_{\mathsf{RLD}}^{A,\mathrm{Src}\,f} = \bigsqcap^{\mathsf{RLD}}_{F\in\mathrm{up}\,f}(\uparrow^{\mathbf{Rel}(\mathrm{Dst}\,f,B)}\mathrm{id}_{\mathrm{Dst}\,f\cap B}\circ F\circ\uparrow^{\mathbf{Rel}(A,\mathrm{Src}\,f)}\mathrm{id}_{A\cap\mathrm{Src}\,f}) = \bigsqcap^{\mathsf{RLD}}_{F\in\mathrm{up}\,f}(\uparrow^{\mathbf{Rel}(A,B)}(\mathrm{id}_{\mathrm{Dst}\,f\cap B}\circ\mathrm{GR}\,F\circ\mathrm{id}_{\mathrm{Dst}\,f\cap B})) = \bigsqcap^{\mathsf{RLD}}_{F\in\mathrm{up}\,f}\uparrow^{\mathbf{Rel}(A,B)}(F\cap(A\times B)) = f\div(A\times B)$. □

PROPOSITION 2143. $\mathrm{id}_X^{\mathsf{RLD}(A,A)} = \mathrm{id}^{\mathsf{RLD}}_{X\div A}$ whenever $A\in\mathfrak{Z}$ and $\mathfrak{A}\ni X\sqsubseteq[A]$.

PROOF. $\mathrm{id}_X^{\mathsf{RLD}(A,A)} = \mathrm{id}^{\mathsf{RLD}}_{X\div(A\cap A)}\div(A\times A) = \mathrm{id}^{\mathsf{RLD}}_{X\div A}$. □

DEFINITION 2144. Category RLD can be considered as a category with binary product morphism with the binary product defined as: $\mathcal{X}\times_{A,B}\mathcal{Y} = (\mathcal{X}\div A)\times^{\mathsf{RLD}}(\mathcal{Y}\div B)$ for every unfixed filters $\mathcal{X}$ and $\mathcal{Y}$.

It is really a binary product morphism:

PROOF. Need to prove the axioms:

1°. $f\sqcap(X\times_{A,B}Y) = f\sqcap((X\div A)\times^{\mathsf{RLD}}(Y\div B)) = \mathrm{id}^{\mathsf{RLD}}_{Y\div B}\circ f\circ\mathrm{id}^{\mathsf{RLD}}_{X\div A} = \mathrm{id}_Y^{\mathsf{RLD}(B,B)}\circ f\circ\mathrm{id}_X^{\mathsf{RLD}(A,A)}$.

2°. Let unfixed filters $X\sqsubseteq[A_0]\sqcap[A_1]$ and $Y\sqsubseteq[B_0]\sqcap[B_1]$. Then we have $\iota_{A_1,B_1}(X\times_{A_0,B_0}Y) = \mathcal{E}^{\mathcal{C}(B_0,B_1)}\circ(X\times_{A_0,B_0}Y)\circ\mathcal{E}^{\mathcal{C}(A_1,A_0)} = \uparrow^{\mathsf{RLD}(B_0,B_1)}\mathrm{id}_{B_0\cap B_1}\circ((X\div A_0)\times^{\mathsf{RLD}}(Y\div B_0))\circ\uparrow^{\mathsf{RLD}(A_1,A_0)}\mathrm{id}_{A_0\cap A_1}$.

But $(X\div A_0)\times^{\mathsf{RLD}}(Y\div B_0) = \bigsqcap^{\mathsf{RLD}}_{x\in\mathrm{up}(X\div A_0),y\in\mathrm{up}(Y\div B_0)}(x\times y) =$ (by the bijection) $= \bigsqcap^{\mathsf{RLD}}_{x\in\mathrm{up}\,X,y\in\mathrm{up}\,Y}((x\div A_0)\times(y\div B_0))$.

Thus by definition of reloidal product $\iota_{A_1,B_1}(X\times_{A_0,B_0}Y) = \bigsqcap^{\mathsf{RLD}(A_1,B_1)}_{x\in\mathrm{up}\,X,y\in\mathrm{up}\,Y}(\mathrm{id}_{B_0\cap B_1}\circ((x\div A_0)\times(y\div B_0))\circ\mathrm{id}_{A_0\cap A_1}) = \bigsqcap^{\mathsf{RLD}(A_1,B_1)}_{x\in\mathrm{up}\,X,y\in\mathrm{up}\,Y}((x\div A_0)\times(y\div B_0)) = \bigsqcap^{\mathsf{RLD}(A_1,B_1)}_{x\in\mathrm{up}(X\div A_0),y\in\mathrm{up}(Y\div B_0)}(x\times y) = (X\div A_1)\times^{\mathsf{RLD}}(Y\div B_1) = X\times_{A_1,B_1}Y$.

□

DEFINITION 2145. Reloid $\mathscr{S}f\in\mathrm{End}_{\mathsf{RLD}}(\text{small sets})$ is defined by the formula $\mathrm{GR}\,\mathscr{S}f = \mathscr{S}\,\mathrm{GR}\,f$ for every reloid f. FiXme: Small sets vs small sets × small sets.

DEFINITION 2146. Reloid $\mathscr{S}f\in\mathrm{End}_{\mathsf{RLD}}(\text{small sets})$ if defined by the formula $\mathscr{S}f = \mathscr{S}F$ for arbitrary $F\in f$ for every unfixed reloid f.

That the result does not depend on the choice of F obviously follows from the corresponding result for filters.

PROPOSITION 2147.

1°. $\mathscr{S}$ from a Hom-set $\mathsf{RLD}(A, B)$ to $\mathrm{End}_{\mathsf{RLD}}(\text{small sets})$ is an order embedding.
2°. $\mathscr{S}$ from the category RLD to $\mathrm{End}_{\mathsf{RLD}}(\text{small sets})$ is a prefunctor.
3°. $\mathscr{S}$ from unfixed reloids is an order embedding and a prefunctor (= semigroup homomorphism).

PROOF.

1°. That it's monotone is obvious. That it is an injection follows from $\mathscr{S}$ for filters being an injection.

2°. Let f and g be composable reloids.

If $H \in \operatorname{up} \mathscr{S}(g \circ f)$ then $H \supseteq H' \in \operatorname{up}(g \circ f)$, $H' \supseteq G \circ F$ for some H', $F \in \operatorname{up} f$ and $G \in \operatorname{up} g$. Consequently $F \in \operatorname{GR} \mathscr{S} f$, $G \in \operatorname{GR} \mathscr{S} g$. So $G \circ F \in \operatorname{up}(\mathscr{S} g \circ \mathscr{S} f)$ and thus $\mathscr{S}(g \circ f) \sqsupseteq \mathscr{S} g \circ \mathscr{S} f$.

Whenever $H \in \operatorname{up}(\mathscr{S} g \circ \mathscr{S} f)$, we have $H \supseteq G \circ F$ where $F \in \operatorname{up} \mathscr{S} f$, $G \in \operatorname{up} \mathscr{S} g$. Thus $F \supseteq F' \in \operatorname{up} f$, $G \supseteq G' \in \operatorname{up} g$; $H \supseteq G' \circ F' \in \operatorname{up}(g \circ f)$ for some F', G' and so $H \in \operatorname{up}(\mathscr{S}(g \circ f))$. So $\mathscr{S} g \circ \mathscr{S} f \sqsupseteq \mathscr{S}(g \circ f)$.

So $\mathscr{S}(g \circ f) = \mathscr{S} g \circ \mathscr{S} f$.

3°. That it is a prefunctor easily follows from the above.

Suppose f, g are unfixed reloids and $\mathscr{S} f = \mathscr{S} g$. Let $F \in f$, $G \in g$ and thus $\mathscr{S} F = \mathscr{S} G$. It is enough to prove that $F \sim G$.

Really, $\mathscr{S} F = \mathscr{S} G \Rightarrow \mathscr{S} \operatorname{GR} F = \mathscr{S} \operatorname{GR} G \Rightarrow \operatorname{GR} F \sim \operatorname{GR} G \Rightarrow \operatorname{GR} G = (\operatorname{GR} F) \div (\operatorname{dom} G \times \operatorname{im} G) \Leftrightarrow G = F \div (\operatorname{dom} G \times \operatorname{im} G) = \iota_{\operatorname{dom} G, \operatorname{im} G} F$. Similarly $F = \iota_{\operatorname{dom} F, \operatorname{im} F} G$. So $F \sim G$.

□

I yet failed to generalize propositions 2137 and 2138. The generalization may require first research pointfree reloids.

8. More results on restricted identities

In the next three propositions assume $A \in \mathfrak{Z}$, $\mathfrak{A} \ni X \sqsubseteq A$.

PROPOSITION 2148. $\mathrm{id}_X^{\mathbf{Rel}(A)} = \mathrm{id}_{[X]}^{\mathbf{Rel}(A,A)}$.

PROOF. $\mathrm{id}_{[X]}^{\mathbf{Rel}(A,A)} = \mathrm{id}_X^{\mathbf{Rel}(A,A)} = \mathrm{id}_X^{\mathbf{Rel}(A)}$. □

PROPOSITION 2149. $\mathrm{id}_X^{\mathsf{FCD}(A)} = \mathrm{id}_{[X]}^{\mathsf{FCD}(A,A)}$.

PROOF. $\left\langle \mathrm{id}_{[X]}^{\mathsf{FCD}(A,A)} \right\rangle \mathcal{X} = ([\mathcal{X}] \sqcap [X]) \div A = [\mathcal{X} \sqcap X]) \div A = \mathcal{X} \sqcap X = \left\langle \mathrm{id}_X^{\mathsf{FCD}(A)} \right\rangle \mathcal{X}$ for $\mathfrak{A} \ni \mathcal{X} \sqsubseteq A$. □

PROPOSITION 2150. $\mathrm{id}_X^{\mathsf{RLD}(A)} = \mathrm{id}_{[X]}^{\mathsf{RLD}(A,A)}$.

PROOF. $\mathrm{id}_{[X]}^{\mathsf{RLD}(A,A)} = \mathrm{id}_{[X] \div (A \cap A)}^{\mathsf{RLD}} \div (A \times A) = \mathrm{id}_X^{\mathsf{RLD}} \div (A \times A) = \mathrm{id}_X^{\mathsf{RLD}(A)}$. □

As a generalization of three last propositions, define for every category $\mathcal{C}$ with restricted identities:

DEFINITION 2151. $\mathrm{id}_X^{\mathcal{C}(A)} = \mathrm{id}_{[X]}^{\mathcal{C}(A,A)}$.

PROPOSITION 2152. $\left\{ \frac{(\mathcal{A} \div A, \mathcal{A} \sqcap A)}{\mathcal{A} \in \mathfrak{F}(U)} \right\}$ is a function and moreover is an order isomorphism for a set $A \subseteq U$.

PROOF. $\mathcal{A} \div A$ and $\mathcal{A} \sqcap A$ are determined by each other by the following formulas:

$$\mathcal{A} \div A = (\mathcal{A} \sqcap A) \div A \quad \text{and} \quad \mathcal{A} \sqcap A = (\mathcal{A} \div A) \div \operatorname{Base}(\mathcal{A}).$$

Prove the formulas: $(\mathcal{A} \sqcap A) \div A = \bigsqcap\left\{\frac{\uparrow^A (X\cap A)}{X\in\mathcal{A}\sqcap A}\right\} = \bigsqcap\left\{\frac{\uparrow^A (X\cap A)}{X\in\mathcal{A}}\right\} = \mathcal{A} \div A$.

$(\mathcal{A} \div A) \div \operatorname{Base}(\mathcal{A}) = \bigsqcap\left\{\frac{\uparrow^A (X\cap A)}{X\in\mathcal{A}}\right\} \div \operatorname{Base}(\mathcal{A}) = \bigsqcap\left\{\frac{\uparrow^{\operatorname{Base}(\mathcal{A})}(Y\cap\operatorname{Base}(\mathcal{A}))}{Y\in\bigsqcap\left\{\frac{\uparrow^A (X\cap A)}{X\in\mathcal{A}}\right\}}\right\} =$ (by properties of filter bases) $= \bigsqcap\left\{\frac{\uparrow^{\operatorname{Base}(\mathcal{A})}(X\cap A\cap\operatorname{Base}(\mathcal{A}))}{X\in}\right\} = \bigsqcap\left\{\frac{\uparrow^{\operatorname{Base}(\mathcal{A})}(X\cap A)}{X\in\mathcal{A}}\right\} = \mathcal{A} \sqcap A$.

That this defines a bijection, follows from $\mathcal{A} \div A \sim \mathcal{A} \sqcap A$ what easily follows from the above formulas. □

PROPOSITION 2153. $\left\{\frac{(\iota_{X,Y} f, \mathrm{id}_Y^{\mathbf{Rel}(\operatorname{Dst} f)} \circ f \circ \mathrm{id}_X^{\mathbf{Rel}(\operatorname{Src} f)})}{f\in\mathbf{Rel}(A,B)}\right\}$ is a function and moreover is an (order and semigroup) isomorphism, for sets $X \subseteq \operatorname{Src} f$, $Y \subseteq \operatorname{Dst} f$.

PROOF. $\iota_{X,Y} f = (X, Y, \operatorname{GR} f \cap (X\times Y))$; $\mathrm{id}_Y^{\mathbf{Rel}} \circ f \circ \mathrm{id}_X^{\mathbf{Rel}} = (\operatorname{Src} f, \operatorname{Dst} f, \operatorname{GR} f \cap (X \times Y))$. The isomorphism (both order and semigroup) is evident. □

PROPOSITION 2154. $\left\{\frac{(\iota_{X,Y} f, \mathrm{id}_Y^{\mathsf{FCD}(\operatorname{Dst} f)} \circ f \circ \mathrm{id}_X^{\mathsf{FCD}(\operatorname{Src} f)})}{f\in\mathsf{FCD}(A,B)}\right\}$ is a function and moreover is an (order and semigroup) isomorphism, for sets $X \subseteq \operatorname{Src} f$, $Y \subseteq \operatorname{Dst} f$.

PROOF. From symmetry it follows that it's enough to prove that $\left\{\frac{(\mathcal{E}^Y \circ f, \mathrm{id}_Y^{\mathsf{FCD}} \circ f)}{f\in\mathsf{FCD}(A,B)}\right\}$ is a function and moreover is an (order and semigroup) isomorphism, for a set $Y \subseteq \operatorname{Dst} f$.

Really, $\left\{\frac{(\langle\mathcal{E}^Y\rangle x, \langle\mathrm{id}_Y^{\mathsf{FCD}}\rangle x)}{x\in\operatorname{Dst} f}\right\} = \left\{\frac{(x\div Y, x\sqcap Y)}{x\in\operatorname{Dst} f}\right\}$ is an order isomorphism by proved above. This implies that $\left\{\frac{(\mathcal{E}^Y \circ f, \mathrm{id}_Y^{\mathsf{FCD}} \circ f)}{f\in\mathsf{FCD}(A,B)}\right\}$ is an isomorphism (both order and semigroup). □

PROPOSITION 2155. $\left\{\frac{(\iota_{X,Y} f, \mathrm{id}_Y^{\mathsf{RLD}(\operatorname{Dst} f)} \circ f \circ \mathrm{id}_X^{\mathsf{RLD}(\operatorname{Src} f)})}{f\in\mathsf{RLD}(A,B)}\right\}$ is a function and moreover is an (order and semigroup) isomorphism, for sets $X \subseteq \operatorname{Src} f$, $Y \subseteq \operatorname{Dst} f$.

PROOF. $\iota_{X,Y} f = (X, Y, (\operatorname{up} f) \div (X \times Y))$; $\mathrm{id}_Y^{\mathsf{RLD}} \circ f \circ \mathrm{id}_X^{\mathsf{RLD}} = (\operatorname{Src} f, \operatorname{Dst} f, (\operatorname{up} f) \sqcap (X \times Y))$. They are order isomorphic by proved above.

$\iota_{Y,Z} g \circ \iota_{X,Y} f = \mathcal{E}^{\operatorname{Dst} g, Z} \circ g \circ \mathcal{E}^{Y, \operatorname{Src} g} \circ \mathcal{E}^{\operatorname{Dst} f, Y} \circ f \circ \mathcal{E}^{X, \operatorname{Src} f} = \mathcal{E}^{\operatorname{Dst} g, Z} \circ g \circ \mathrm{id}_Y^{\mathsf{RLD}} \circ \mathrm{id}_Y^{\mathsf{RLD}} \circ f \circ \mathcal{E}^{X, \operatorname{Src} f}$ because $\mathcal{E}^{Y, \operatorname{Src} g} \circ \mathcal{E}^{\operatorname{Dst} f, Y} = \mathrm{id}_Y^{\mathbf{Rel}} = \mathrm{id}_Y^{\mathbf{Rel}} \circ \mathrm{id}_Y^{\mathbf{Rel}}$. Thus by proved above

$$\left\{\frac{(\iota_{Y,Z} g \circ \iota_{X,Y} f, \mathrm{id}_Z^{\mathsf{RLD}} \circ g \circ \mathrm{id}_Y^{\mathsf{RLD}} \circ \mathrm{id}_Y^{\mathsf{RLD}} \circ f \circ \mathrm{id}_X^{\mathsf{RLD}})}{f \in \mathsf{RLD}(A, B)}\right\}$$

is a bijection. □

Can three last propositions be generalized into one?

CHAPTER 3

Applications of algebraic general topology

1. "Hybrid" objects

Algebraic general topology allows to construct "hybrid" objects of "continuous" (as topological spaces) and discrete (as graphs).

Consider for example $D \sqcup T$ where D is a digraph and T is a topological space.

The n-th power $(D \sqcup T)^n$ yields an expression with 2^n terms. So treating $D \sqcup T$ as one object (what becomes possible using algebraic general topology) rather than the join of two objects may have an exponential benefit for simplicity of formulas.

2. A way to construct directed topological spaces

2.1. Some notation. I use $\mathcal{E}$ and ι notations from `volume-2.pdf`. FiXme: Reorder document fragments to describe it before use.

I remind that $f|_X = f \circ \mathrm{id}_X$ for binary relations, funcoids, and reloid.

$f \parallel_X = f \circ (\mathcal{E}^X)^{-1}$.

$f \square X = \mathrm{id}_X \circ f \circ \mathrm{id}_X^{-1}$.

As proved in `volume-2.pdf`, the following are bijections and moreover isomorphisms (for R being either funcoids or reloids or binary relations):

1°. $\left\{ \frac{(f|_X, f\parallel_X)}{f \in R} \right\}$;
2°. $\left\{ \frac{(f \square X, \iota_X f)}{f \in R} \right\}$.

As easily follows from these isomorphisms and theorem 1285:

PROPOSITION 2156. For funcoids, reloids, and binary relations:

1°. $f \in \mathrm{C}(\mu, \nu) \Rightarrow f \parallel_A \in \mathrm{C}(\iota_A \mu, \nu)$;
2°. $f \in \mathrm{C}'(\mu, \nu) \Rightarrow f \parallel_A \in \mathrm{C}'(\iota_A \mu, \nu)$;
3°. $f \in \mathrm{C}''(\mu, \nu) \Rightarrow f \parallel_A \in \mathrm{C}''(\iota_A \mu, \nu)$.

2.2. Directed line and directed intervals. Let $\mathfrak{A}$ be a poset. We will denote $\overline{\mathfrak{A}} = \mathfrak{A} \cup \{-\infty, +\infty\}$ the poset with two added elements $-\infty$ and $+\infty$, such that $+\infty$ is strictly greater than every element of $\mathfrak{A}$ and $-\infty$ is strictly less.

FiXme: Generalize from $\mathbb{R}$ to a wider class of posets.

DEFINITION 2157. For an element a of a poset $\mathfrak{A}$

1°. $J_{\geq}(a) = \left\{ \frac{x \in \mathfrak{A}}{x \geq a} \right\}$;
2°. $J_{>}(a) = \left\{ \frac{x \in \mathfrak{A}}{x > a} \right\}$;
3°. $J_{\leq}(a) = \left\{ \frac{x \in \mathfrak{A}}{x \leq a} \right\}$;
4°. $J_{<}(a) = \left\{ \frac{x \in \mathfrak{A}}{x < a} \right\}$;
5°. $J_{\neq}(a) = \left\{ \frac{x \in \mathfrak{A}}{x \neq a} \right\}$.

DEFINITION 2158. Let a be an element of a poset $\mathfrak{A}$.

1°. $\Delta(a) = \bigsqcap^{\mathscr{F}} \left\{ \frac{]x;y[}{x,y \in \overline{\mathfrak{A}}, x < a \wedge y > a} \right\}$;
2°. $\Delta_{\geq}(a) = \bigsqcap^{\mathscr{F}} \left\{ \frac{[a;y[}{y \in \overline{\mathfrak{A}}, y > a} \right\}$;

3°. $\Delta_{>}(a) = \prod^{\mathscr{F}} \left\{ \frac{]a;y[}{y \in \overline{\mathfrak{A}}, x < a \wedge y > a} \right\}$;
4°. $\Delta_{\leq}(a) = \prod^{\mathscr{F}} \left\{ \frac{]x;a]}{x \in \overline{\mathfrak{A}}, x < a} \right\}$;
5°. $\Delta_{<}(a) = \prod^{\mathscr{F}} \left\{ \frac{]x;a[}{x \in \overline{\mathfrak{A}}, x < a} \right\}$;
6°. $\Delta_{\neq}(a) = \Delta(a) \setminus \{a\}$.

OBVIOUS 2159.
1°. $\Delta_{\geq}(a) = \Delta(a) \sqcap^{\mathscr{F}} @J_{\geq}(a)$;
2°. $\Delta_{>}(a) = \Delta(a) \sqcap^{\mathscr{F}} @J_{>}(a)$;
3°. $\Delta_{\leq}(a) = \Delta(a) \sqcap^{\mathscr{F}} @J_{\leq}(a)$;
4°. $\Delta_{<}(a) = \Delta(a) \sqcap^{\mathscr{F}} @J_{<}(a)$;
5°. $\Delta_{\neq}(a) = \Delta(a) \sqcap^{\mathscr{F}} @J_{\neq}(a)$.

DEFINITION 2160. Given a partial order $\mathfrak{A}$ and $x \in \mathfrak{A}$, the following defines complete funcoids:
1°. $\langle|\mathfrak{A}|\rangle^* \{x\} = \Delta(x)$;
2°. $\langle|\mathfrak{A}|_{\geq}\rangle^* \{x\} = \Delta_{\geq}(x)$;
3°. $\langle|\mathfrak{A}|_{>}\rangle^* \{x\} = \Delta_{>}(x)$;
4°. $\langle|\mathfrak{A}|_{\leq}\rangle^* \{x\} = \Delta_{\leq}(x)$;
5°. $\langle|\mathfrak{A}|_{<}\rangle^* \{x\} = \Delta_{<}(x)$;
6°. $\langle|\mathfrak{A}|_{\neq}\rangle^* \{x\} = \Delta_{\neq}(x)$.

PROPOSITION 2161. The complete funcoid corresponding to the order topology[1] is equal to $|\mathfrak{A}|$.

PROOF. Because every open set is a finite union of open intervals, the complete funcoid f corresponding to the order topology is described by the formula: $\langle f\rangle^* \{x\} = \prod^{\mathscr{F}} \left\{ \frac{]a;b[}{a,b \in \overline{\mathfrak{A}}, a < x \wedge b > x} \right\} = \Delta(x) = \langle|\mathfrak{A}|\rangle^* \{x\}$. Thus $f = |\mathfrak{A}|$. □

EXERCISE 2162. Show that $|\mathfrak{A}|_{\geq}$ (in general) is not the same as "right order topology"[2].

PROPOSITION 2163.
1°. $\left\langle|\mathfrak{A}|_{\geq}^{-1}\right\rangle^* @X = @\left\{ \frac{a \in \mathfrak{A}}{\forall y \in \overline{\mathfrak{A}} : (y > a \Rightarrow X \cap [a;y[\neq \emptyset)} \right\}$;
2°. $\left\langle|\mathfrak{A}|_{>}^{-1}\right\rangle^* @X = @\left\{ \frac{a \in \mathfrak{A}}{\forall y \in \overline{\mathfrak{A}} : (y > a \Rightarrow X \cap]a;y[\neq \emptyset)} \right\}$;
3°. $\left\langle|\mathfrak{A}|_{\leq}^{-1}\right\rangle^* @X = @\left\{ \frac{a \in \mathfrak{A}}{\forall x \in \overline{\mathfrak{A}} : (x < a \Rightarrow X \cap]x;a] \neq \emptyset)} \right\}$;
4°. $\left\langle|\mathfrak{A}|_{<}^{-1}\right\rangle^* @X = @\left\{ \frac{a \in \mathfrak{A}}{\forall x \in \overline{\mathfrak{A}} : (x < a \Rightarrow X \cap]x;a[\neq \emptyset)} \right\}$.

PROOF. $a \in \left\langle|\mathfrak{A}|_{\geq}^{-1}\right\rangle^* @X \Leftrightarrow @\{a\} \not\asymp \left\langle|\mathfrak{A}|_{\geq}^{-1}\right\rangle^* @X \Leftrightarrow \langle|\mathfrak{A}|_{\geq}\rangle^* @\{a\} \not\asymp @X \Leftrightarrow \Delta_{\geq}(a) \not\asymp @X \Leftrightarrow \forall y \in \overline{\mathfrak{A}} : (y > a \Rightarrow X \cap [a; y[\neq \emptyset)$.

$a \in \left\langle|\mathfrak{A}|_{>}^{-1}\right\rangle^* @X \Leftrightarrow @\{a\} \not\asymp \left\langle|\mathfrak{A}|_{>}^{-1}\right\rangle^* @X \Leftrightarrow \langle|\mathfrak{A}|_{>}\rangle^* @\{a\} \not\asymp @X \Leftrightarrow \Delta_{>}(a) \not\asymp @X \Leftrightarrow \forall y \in \overline{\mathfrak{A}} : (y > a \Rightarrow X \cap]a; y[\neq \emptyset)$.

The rest follows from duality. □

REMARK 2164. On trivial ultrafilters these obviously agree:
1°. $\langle|\mathbb{R}|_{\geq}\rangle^* \{x\} = \langle|\mathbb{R}| \sqcap \geq\rangle^* \{x\}$;
2°. $\langle|\mathbb{R}|_{>}\rangle^* \{x\} = \langle|\mathbb{R}| \sqcap >\rangle^* \{x\}$;
3°. $\langle|\mathbb{R}|_{\leq}\rangle^* \{x\} = \langle|\mathbb{R}| \sqcap \leq\rangle^* \{x\}$;
4°. $\langle|\mathbb{R}|_{<}\rangle^* \{x\} = \langle|\mathbb{R}| \sqcap <\rangle^* \{x\}$.

[1]See Wikipedia for a definition of "Order topology".
[2]See Wikipedia

COROLLARY 2165.

1°. $|\mathbb{R}|_{\geq} = \operatorname{Compl}(|\mathbb{R}| \sqcap \geq)$;
2°. $|\mathbb{R}|_{>} = \operatorname{Compl}(|\mathbb{R}| \sqcap >)$;
3°. $|\mathbb{R}|_{\leq} = \operatorname{Compl}(|\mathbb{R}| \sqcap \leq)$;
4°. $|\mathbb{R}|_{<} = \operatorname{Compl}(|\mathbb{R}| \sqcap <)$.

OBVIOUS 2166. FiXme: also what is the values of $\setminus$ operation

1°. $|\mathbb{R}|_{\geq} = |\mathbb{R}|_{>} \sqcup 1$;
2°. $|\mathbb{R}|_{\leq} = |\mathbb{R}|_{<} \sqcup 1$.

3. Some inequalities

FiXme: Define the ultrafilter "at the left" and "at the right" of a real number. Also define "convergent ultrafilter".

Denote $\Delta_{+\infty} = \bigsqcap_{x\in\mathbb{R}}]x; +\infty[$ and $\Delta_{-\infty} = \bigsqcap_{x\in\mathbb{R}}] - \infty; x[$.

The following proposition calculates $\langle\geq\rangle x$ and $\langle>\rangle x$ for all kinds of ultrafilters on $\mathbb{R}$:

PROPOSITION 2167.

1°. $\langle\geq\rangle\{\alpha\} = [\alpha; +\infty[$ and $\langle>\rangle\{\alpha\} =]\alpha; +\infty[$.
2°. $\langle\geq\rangle x = \langle>\rangle x =]\alpha; +\infty[$ for ultrafilter x at the right of a number α.
3°. $\langle\geq\rangle x = \langle>\rangle x = \Delta_{<}(\alpha) \sqcup [\alpha; +\infty[= \Delta_{\leq}(\alpha) \sqcup]\alpha; +\infty[$ for ultrafilter x at the left of a number α.
4°. $\langle\geq\rangle x = \langle>\rangle x = \Delta_{+\infty}$ for ultrafilter x at positive infinity.
5°. $\langle\geq\rangle x = \langle>\rangle x = \mathbb{R}$ for ultrafilter x at negative infinity.

PROOF.
1°. Obvious.
2°.

$$\langle\geq\rangle x = \bigsqcap^{\mathscr{F}}_{X\in\operatorname{up} x} \langle\geq\rangle(X \sqcap]\alpha; +\infty[) = \bigsqcap^{\mathscr{F}}_{X\in\operatorname{up} x}]\alpha; +\infty[=]\alpha; +\infty[;$$

$$\langle>\rangle x = \bigsqcap^{\mathscr{F}}_{X\in\operatorname{up} x} \langle>\rangle(X \sqcap]\alpha; +\infty[) = \bigsqcap^{\mathscr{F}}_{X\in\operatorname{up} x}]\alpha; +\infty[=]\alpha; +\infty[.$$

3°. $\Delta_{<}(\alpha) \sqcup [\alpha; +\infty[= \Delta_{\leq}(\alpha) \sqcup]\alpha; +\infty[$ is obvious.

$$\langle>\rangle x = \bigsqcap^{\mathscr{F}}_{X\in\operatorname{up} x} \langle>\rangle X \sqsupseteq \bigsqcap^{\mathscr{F}}_{X\in\operatorname{up} x} (\Delta_{<}(\alpha) \sqcup]\alpha; +\infty[) = \Delta_{<}(\alpha) \sqcup]\alpha; +\infty[$$

but $\langle\geq\rangle x \sqsubseteq \Delta_{<}(\alpha) \sqcup [\alpha; +\infty[$ is obvious. It remains to take into account that $\langle>\rangle x \sqsubseteq \langle\geq\rangle x$.

4°. $\langle\geq\rangle x = \bigsqcap^{\mathscr{F}}_{X\in\operatorname{up} x} \langle\geq\rangle X = \bigsqcap^{\mathscr{F}}_{X\in\operatorname{up} x, \inf X \in X} \langle\geq\rangle(X \sqcap]\alpha; +\infty[) = \bigsqcap^{\mathscr{F}}_{X\in\operatorname{up} x} [\inf X; +\infty[= \bigsqcap^{\mathscr{F}}_{x>\alpha} [x; +\infty[= \Delta_{+\infty}$; $\langle>\rangle x = \bigsqcap^{\mathscr{F}}_{X\in\operatorname{up} x} \langle>\rangle X = \bigsqcap^{\mathscr{F}}_{X\in\operatorname{up} x, \inf X\in X} \langle>\rangle(X \sqcap]\alpha; +\infty[) = \bigsqcap^{\mathscr{F}}_{X\in\operatorname{up} x}]\inf X; +\infty[= \bigsqcap^{\mathscr{F}}_{x>\alpha} [x; +\infty[= \Delta_{+\infty}$.

5°. $\langle\geq\rangle x \sqsupseteq \langle>\rangle x = \bigsqcap^{\mathscr{F}}_{X\in\operatorname{up} x} \langle>\rangle X$ but $\langle>\rangle X =] - \infty; +\infty[$ for $X \in \operatorname{up} x$ because X has arbitrarily small elements.

□

LEMMA 2168. $\langle|\mathbb{R}|\rangle x \sqsubseteq \langle>\rangle x = \langle\geq\rangle x$ for every nontrivial ultrafilter x.

PROOF. $\langle>\rangle x = \langle\geq\rangle x$ follows from the previous proposition.
$\langle|\mathbb{R}|\rangle x = \bigsqcap_{X\in\operatorname{up} x} \langle|\mathbb{R}|\rangle X = \bigsqcap_{X\in\operatorname{up} x} \bigsqcup_{y\in X} \Delta(y)$.
Consider cases:

x is an ultrafilter at the right of some number α.
$\langle|\mathbb{R}|\rangle x = \bigsqcap_{X\in\operatorname{up}x}\bigsqcup_{y\in X\sqcap]\alpha;+\infty[}\Delta(y) \sqsubseteq]\alpha;+\infty[= \langle\geq\rangle x$ because $\bigsqcup_{y\in X\sqcap]\alpha;+\infty[}\Delta(y) \sqsubseteq]\alpha;+\infty[$.

x is an ultrafilter at the left of some number α.
$\langle|\mathbb{R}|\rangle x \sqsubseteq \Delta(\alpha)$ is obvious. But $\langle\geq\rangle x \sqsupseteq \Delta(\alpha)$.

x is an ultrafilter at positive infinity.
$\langle|\mathbb{R}|\rangle x \sqsubseteq \Delta_{+\infty}$ is obvious. But $\langle\geq\rangle x = \Delta_{+\infty}$.

x is an ultrafilter at negative infinity.
Because $\langle\geq\rangle x = \mathbb{R}$.

□

Corollary 2169. $\langle|\mathbb{R}|\sqcap\geq\rangle x = \langle|\mathbb{R}|\rangle x$ for every nontrivial ultrafilter x.

Proof. $\langle|\mathbb{R}|\sqcap\geq\rangle x = \langle|\mathbb{R}|\rangle\sqcap\langle\geq\rangle x = \langle|\mathbb{R}|\rangle x$. □

So $\langle|\mathbb{R}|\sqcap\geq\rangle$ and $\langle|\mathbb{R}|\rangle$ agree on all ultrafilters except trivial ones.

Proposition 2170. $|\mathbb{R}|_{>}\sqcap > = |\mathbb{R}|_{>}\sqcap \geq = |\mathbb{R}|_{>}$.

Proof. $|\mathbb{R}|_{>} \sqsubseteq >$ because $\langle|\mathbb{R}|_{>}\rangle^{*}x \sqsubseteq \langle>\rangle^{*}x$ and $|\mathbb{R}|_{>}$ is a complete funcoid. □

Lemma 2171. $\langle|\mathbb{R}|_{>}\rangle x \sqsubset \langle|\mathbb{R}|_{\geq}\rangle x$ for a nontrivial ultrafilter x.

Proof. It enough to prove $\langle|\mathbb{R}|_{>}\rangle x \neq \langle|\mathbb{R}|_{\geq}\rangle x$.
Take x be an ultrafilter with limit point 0 on $\operatorname{im} z$ where z is the sequence $n\mapsto\frac{1}{n}$.

$$\langle|\mathbb{R}|_{>}\rangle x \sqsubseteq \langle|\mathbb{R}|_{>}\rangle^{*}\operatorname{im} z = \bigsqcup_{n\in\operatorname{im} z}\Delta_{>}\left(\frac{1}{n}\right) \sqsubseteq \bigsqcup_{n\in\operatorname{im} z}\left]\frac{1}{n};\frac{1}{n-1}-\frac{1}{n}\right[\asymp \operatorname{im} z.$$

Thus $\langle|\mathbb{R}|_{>}\rangle x \asymp \operatorname{im} z$. But $\langle|\mathbb{R}|_{\geq}\rangle x \sqsubseteq \langle=\rangle x \not\asymp \operatorname{im} z$. □

Corollary 2172. $|\mathbb{R}|_{>} \sqsubset |\mathbb{R}|_{\geq}$.

Proposition 2173. $|\mathbb{R}|_{>} \sqsubset |\mathbb{R}|_{\geq}\sqcap >$.

Proof. It's enough to prove $|\mathbb{R}|_{>} \neq |\mathbb{R}|_{\geq}\sqcap >$.
Really, $\langle|\mathbb{R}|_{\geq}\sqcap >\rangle x = \langle|\mathbb{R}|_{\geq}\rangle x \neq \langle|\mathbb{R}|_{>}\rangle x$ (lemma). □

Proposition 2174.
1°. $|\mathbb{R}|_{\geq}\circ|\mathbb{R}|_{\geq} = |\mathbb{R}|_{\geq}$;
2°. $|\mathbb{R}|_{>}\circ|\mathbb{R}|_{>} = |\mathbb{R}|_{>}$;
3°. $|\mathbb{R}|_{\geq}\circ|\mathbb{R}|_{>} = |\mathbb{R}|_{>}$;
4°. $|\mathbb{R}|_{>}\circ|\mathbb{R}|_{\geq} = |\mathbb{R}|_{>}$.

Proof. ?? □

Conjecture 2175.
1°. $(|\mathbb{R}|\sqcap\geq)\circ(|\mathbb{R}|\sqcap\geq) = |\mathbb{R}|\sqcap\geq$.
2°. $(|\mathbb{R}|\sqcap>)\circ(|\mathbb{R}|\sqcap>) = |\mathbb{R}|\sqcap>$.

4. Continuity

I will say that a property holds on a filter $\mathcal{A}$ iff there is $A\in\operatorname{up}\mathcal{A}$ on which the property holds.

FiXme: $f\in\mathrm{C}(A,B)\wedge f\in\mathrm{C}(\iota_A|\mathbb{R}|_{\geq},\iota_B|\mathbb{R}|_{\geq}) \Leftrightarrow (f,f)\in\mathrm{C}((A,\iota_A|\mathbb{R}|_{\geq}),(B,\iota_B|\mathbb{R}|_{\geq}))$

Lemma 2176. Let function $f:A\to B$ where $A,B\in\mathscr{P}\mathbb{R}$ and A is connected.

1°. f is monotone and $f \in \mathrm{C}(A, B)$ iff $f \in \mathrm{C}(A, B) \cap \mathrm{C}(\iota_A|\mathbb{R}|_{\geq}, \iota_B|\mathbb{R}|_{\geq})$ iff $f \in \mathrm{C}(A, B) \cap \mathrm{C}(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{\geq})$ iff $f \in \mathrm{C}(\iota_A|\mathbb{R}|_{\geq}, \iota_B|\mathbb{R}|_{\geq}) \cap \mathrm{C}(\iota_A|\mathbb{R}|_{\leq}, \iota_B|\mathbb{R}|_{\leq})$.

2°. f is strictly monotone and $\in \mathrm{C}(A, B)$ iff $f \in \mathrm{C}(A, B) \cap \mathrm{C}(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{>})$ iff $f \in \mathrm{C}(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{>}) \cap \mathrm{C}(\iota_A|\mathbb{R}|_{<}, \iota_B|\mathbb{R}|_{<})$.

FiXme: Generalize for arbitrary posets. FiXme: Generalize for f being a funcoid.

Proof. Because f is continuous, we have $\langle f \circ \iota_A|\mathbb{R}|\rangle^* \{x\} \sqsubseteq \langle \iota_B|\mathbb{R}| \circ f\rangle^* \{x\}$ that is $\langle f\rangle^* \Delta(x) \sqsubseteq \Delta(f(x))$ for every x.

If f is monotone, we have $\langle f\rangle^* \Delta_{\geq}(x) \sqsubseteq [f(x); \infty[$. Thus $\langle f\rangle^* \Delta_{\geq}(x) \sqsubseteq \Delta_{\geq}(f(x))$, that is $\langle f \circ \iota_A|\mathbb{R}|_{\geq}\rangle^* \{x\} \sqsubseteq \langle \iota_B|\mathbb{R}|_{\geq} \circ f\rangle^* \{x\}$, thus $f \in \mathrm{C}(\iota_A|\mathbb{R}|_{\geq}, \iota_B|\mathbb{R}|_{\geq})$.

If f is strictly monotone, we have $\langle f\rangle^* \Delta_{>}(x) \sqsubseteq]f(x); \infty[$. Thus $\langle f\rangle^* \Delta_{>}(x) \sqsubseteq \Delta_{>}(f(x))$, that is $\langle f \circ \iota_A|\mathbb{R}|_{>}\rangle^* \{x\} \sqsubseteq \langle \iota_B|\mathbb{R}|_{>} \circ f\rangle^* \{x\}$, thus $f \in \mathrm{C}(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{>})$.

Let now $f \in \mathrm{C}(\iota_A|\mathbb{R}|_{\geq}, \iota_B|\mathbb{R}|_{\geq})$.

Take any $a \in A$ and let $c = \left\{ \frac{b \in B}{b \geq a, \forall x \in [a; b[: f(x) \geq f(a)} \right\}$. It's enough to prove that c is the right endpoint (finite or infinite) of A.

Indeed by continuity $f(a) \leq f(c)$ and if c is not already the right endpoint of A, then there is $b' > c$ such that $\forall x \in [c; b'[: f(x) \geq f(c)$. So we have $\forall x \in [a; b'[: f(x) \geq f(c)$ what contradicts to the above.

So f is monotone on the entire A.

$f \in \mathrm{C}(\iota_A|\mathbb{R}|_{\geq}, \iota_B|\mathbb{R}|_{\geq}) \Rightarrow f \in \mathrm{C}(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{\geq})$ is obvious. Reversely $f \in \mathrm{C}(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{\geq}) \Rightarrow f \circ \iota_A|\mathbb{R}|_{>} \sqsubseteq \iota_B|\mathbb{R}|_{\geq} \circ f \Leftrightarrow \forall x \in \mathbb{R} : \langle f\rangle\langle \iota_A|\mathbb{R}|_{>}\rangle^* \{x\} \sqsubseteq \langle \iota_B|\mathbb{R}|_{\geq}\rangle^* \langle f\rangle^* \{x\} \Leftrightarrow \forall x \in \mathbb{R} : \langle f\rangle \Delta_{>}(x) \sqsubseteq \Delta_{\geq} f(x) \Leftrightarrow \forall x \in \mathbb{R} : \langle f\rangle \Delta_{>}(x) \sqcup \{f(x)\} \sqsubseteq \Delta_{\geq} f(x) \Leftrightarrow \forall x \in \mathbb{R} : \langle f\rangle \Delta_{>}(x) \sqcup \{x\} \sqsubseteq \Delta_{\geq} f(x) \Leftrightarrow \forall x \in \mathbb{R} : \langle f\rangle \Delta_{\geq}(x) \sqsubseteq \Delta_{\geq} f(x) \Leftrightarrow \forall x \in \mathbb{R} : \langle f\rangle\langle \iota_A|\mathbb{R}|_{\geq}\rangle^* \{x\} \sqsubseteq \langle \iota_B|\mathbb{R}|_{\geq}\rangle^* \langle f\rangle^* \{x\} \Leftrightarrow \forall x \in \mathbb{R} : f \circ \iota_A|\mathbb{R}|_{\geq} \sqsubseteq \iota_B|\mathbb{R}|_{\geq} \circ f \Leftrightarrow f \in \mathrm{C}(\iota_A|\mathbb{R}|_{\geq}, \iota_B|\mathbb{R}|_{\geq})$.

Let $f \in \mathrm{C}(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{>})$. Then $f \in \mathrm{C}(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{\geq})$ and thus it is monotone. We need to prove that f is strictly monotone. Suppose the contrary. Then there is a nonempty interval $[p; q] \subseteq A$ such that f is constant on this interval. But this is impossible because $f \in \mathrm{C}(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{>})$.

Prove that $f \in \mathrm{C}(\iota_A|\mathbb{R}|_{\geq}, \iota_B|\mathbb{R}|_{\geq}) \cap \mathrm{C}(\iota_A|\mathbb{R}|_{\leq}, \iota_B|\mathbb{R}|_{\leq})$ implies $f \in \mathrm{C}(A, B)$. Really, it implies $\langle f\rangle \Delta_{\leq}(x) \sqsubseteq \Delta_{\leq}(fx)$ and $\langle f\rangle \Delta_{\geq}(x) \sqsubseteq \Delta_{\geq}(fx)$ thus $\langle f\rangle \Delta(x) = \langle f\rangle(\Delta_{\leq}(x) \sqcup \{x\} \sqcup \Delta_{\geq}(x)) \subseteq \Delta_{\leq} f(x) \sqcup \{f(x)\} \sqcup \Delta_{\geq} f(x) = \Delta(f(x))$.

Prove that $f \in \mathrm{C}(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{>}) \cap \mathrm{C}(\iota_A|\mathbb{R}|_{<}, \iota_B|\mathbb{R}|_{<})$ $f \in \mathrm{C}(A, B)$. Really, it implies $\langle f\rangle \Delta_{<}(x) \sqsubseteq \Delta_{<}(fx)$ and $\langle f\rangle \Delta_{>}(x) \sqsubseteq \Delta_{>}(fx)$ thus $\langle f\rangle \Delta(x) = \langle f\rangle(\Delta_{<}(x) \sqcup \{x\} \sqcup \Delta_{>}(x)) \subseteq \Delta_{<} f(x) \sqcup \{f(x)\} \sqcup \Delta_{>} f(x) = \Delta(f(x))$. □

Theorem 2177. Let function $f : A \to B$ where $A, B \in \mathscr{P}\mathbb{R}$.

1°. f is locally monotone and $f \in \mathrm{C}(A, B)$ iff $f \in \mathrm{C}(A, B) \cap \mathrm{C}(\iota_A|\mathbb{R}|_{\geq}, \iota_B|\mathbb{R}|_{\geq})$ iff $f \in \mathrm{C}(A, B) \cap \mathrm{C}(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{\geq})$ iff $f \in \mathrm{C}(\iota_A|\mathbb{R}|_{\geq}, \iota_B|\mathbb{R}|_{\geq}) \cap \mathrm{C}(\iota_A|\mathbb{R}|_{\leq}, \iota_B|\mathbb{R}|_{\leq})$.

2°. f is locally strictly monotone and $\in \mathrm{C}(A, B)$ iff $f \in \mathrm{C}(A, B) \cap \mathrm{C}(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{>})$ iff $f \in \mathrm{C}(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{>}) \cap \mathrm{C}(\iota_A|\mathbb{R}|_{<}, \iota_B|\mathbb{R}|_{<})$.

Proof. By the lemma it is (strictly) monotone on each connected component. □

See also related math.SE questions:

1°. http://math.stackexchange.com/q/1473668/4876
2°. http://math.stackexchange.com/a/1872906/4876
3°. http://math.stackexchange.com/q/1875975/4876

4.1. Directed topological spaces. Directed topological spaces are defined at
http://ncatlab.org/nlab/show/directed+topological+space

Definition 2178. A *directed topological space* (or *d-space* for short) is a pair (X, d) of a topological space X and a set $d \subseteq \mathrm{C}([0;1], X)$ (called *directed paths* or *d-paths*) of paths in X such that

1°. (constant paths) every constant map $[0;1] \to X$ is directed;
2°. (reparameterization) d is closed under composition with increasing continuous maps $[0;1] \to [0;1]$;
3°. (concatenation) d is closed under path-concatenation: if the d-paths a, b are consecutive in X ($a(1) = b(0)$), then their ordinary concatenation $a + b$ is also a d-path

$$(a+b)(t) = a(2t), \text{ if } 0 \leq t \leq \frac{1}{2},$$
$$(a+b)(t) = b(2t-1), \text{ if } \frac{1}{2} \leq t \leq 1.$$

I propose a new way to construct a directed topological space. My way is more geometric/topological as it does not involve dealing with particular paths.

Definition 2179. Let T be the complete endofuncoid corresponding to a topological space and $\nu \sqsubseteq T$ be its "subfuncoid". The d-space $(\mathrm{dir})(T, \nu)$ induced by the pair (T, ν) consists of T and paths $f \in \mathrm{C}([0;1], T) \cap \mathrm{C}(|[0;1]|_{\geq}, \nu)$ such that $f(0) = f(1)$.

Proposition 2180. It is really a d-space.

Proof. Every d-path is continuous.

Constant path are d-paths because ν is reflexive.

Every reparameterization is a d-path because they are $\mathrm{C}(|[0;1]|_{\geq}, \nu)$ and we can apply the theorem about composition of continuous functions.

Every concatenation is a d-path. Denote $f_0 = \lambda t \in [0; \frac{1}{2}] : a(2t)$ and $f_1 = \lambda t \in [\frac{1}{2}; 1] : b(2t-1)$. Obviously $f_0, f_1 \in \mathrm{C}([0;1], \mu) \cap \mathrm{C}(|[0;1]|_{\geq}, \nu)$. Then we conclude that $a + b = f_1 \sqcup f_1$ is in $f_0, f_1 \in \mathrm{C}([0;1], \mu) \cap \mathrm{C}(|[0;1]|_{\geq}, \nu)$ using the fact that the operation $\circ$ is distributive over $\sqcup$. □

Below we show that not every d-space is induced by a pair of an endofuncoid and its subfuncoid. But are d-spaces not represented this way good anything except counterexamples?

Let now we have a d-space (X, d). Define funcoid ν corresponding to the d-space by the formula $\nu = \bigsqcup_{a \in d}(a \circ |\mathbb{R}|_{\geq} \circ a^{-1})$.

Example 2181. The two directed topological spaces, constructed from a fixed topological space and two different reflexive funcoids, are the same.

Proof. Consider the indiscrete topology T on $\mathbb{R}$ and the funcoids $1^{\mathsf{FCD}(\mathbb{R},\mathbb{R})}$ and $1^{\mathsf{FCD}(\mathbb{R},\mathbb{R})} \sqcup (\{0\} \times^{\mathsf{FCD}} \Delta_{\geq})$. The only d-paths in both these settings are constant functions. □

Example 2182. A d-space is not determined by the induced funcoid.

Proof. The following a d-space induces the same funcoid as the d-space of all paths on the plane.

Consider a plane $\mathbb{R}^2$ with the usual topology. Let d-paths be paths lying inside a polygonal chain (in the plane). □

CONJECTURE 2183. A d-path a is determined by the funcoids (where x spans $[0;1]$)

$$(\lambda t \in \mathbb{R} : a(x+t))|_{\Delta(0)}.$$

5. A way to construct directed topological spaces

I propose a new way to construct a directed topological space. My way is more geometric/topological as it does not involve dealing with particular paths.

CONJECTURE 2184. Every directed topological space can be constructed in the below described way.

Consider topological space T and its subfuncoid F (that is F is a funcoid which is less that T in the order of funcoids). Note that in our consideration F is an endofuncoid (its source and destination are the same).

Then a directed path from point A to point B is defined as a continuous function f from $[0;1]$ to F such that $f(0) = A$ and $f(1) = B$. FiXme: Specify whether the interval $[0;1]$ is treated as a proximity, pretopology, or preclosure.

Because F is less that T, we have that every directed path is a path.

CONJECTURE 2185. The two directed topological spaces, constructed from a fixed topological space and two different funcoids, are different.

For a counter-example of (which of the two?) the conjecture consider funcoid $T \sqcap (\mathbb{Q} \times^{\mathsf{FCD}} \mathbb{Q})$ where T is the usual topology on real line.We need to consider stability of existence and uniqueness of a path under transformations of our funcoid and under transformations of the vector field. Can this be a step to solve Navier-Stokes existence and smoothness problems?

6. Integral curves

We will consider paths in a normed vector space V.

DEFINITION 2186. Let D be a connected subset of $\mathbb{R}$. A *path* is a function $D \to V$.

Let d be a vector field in a normed vector space V.

DEFINITION 2187. *Integral curve* of a vector field d is a differentiable function $f : D \to V$ such that $f'(t) = d(f(t))$ for every $t \in D$.

DEFINITION 2188. The definition of *right side integral curve* is the above definition with right derivative of f instead of derivative f'. *Left side integral curve* is defined similarly.

6.1. Path reparameterization.

C^1 is a function which has continuous derivative on every point of the domain.

By D^1 I will denote a C^1 function whose derivative is either nonzero at every point or is zero everywhere.

DEFINITION 2189. A *reparameterization* of a C^1 path is a bijective C^1 function $\phi : D \to D$ such that $\phi'(t) > 0$. A curve f_2 is called a reparametrized curve f_1 if there is a reparameterization ϕ such that $f_2 = f_1 \circ \phi$.

It is well known that this defines an equivalence relation of functions.

PROPOSITION 2190. Reparameterization of D^1 function is D^1.

PROOF. If the function has zero derivative, it is obvious.

Let f_1 has everywhere nonzero derivative. Then $f_2'(t) = f_1'(\phi(t))\phi'(t)$ what is trivially nonzero. □

DEFINITION 2191. Vectors p and q have the *same direction* ($p \upuparrows q$) iff there exists a strictly positive real c such that $p = cq$.

OBVIOUS 2192. Being same direction is an equivalence relation.

OBVIOUS 2193. The only vector with the same direction as the zero vector is zero vector.

THEOREM 2194. A D^1 function y is some reparameterization of a D^1 integral curve x of a continuous vector field d iff $y'(t) \upuparrows d(y(t))$ that is vectors $y'(t)$ and $d(y(t))$ have the same direction (for every t).

PROOF. If y is a reparameterization of x, then $y(t) = x(\phi(t))$. Thus $y'(t) = x'(\phi(t))\phi'(t) = d(x(\phi(t)))\phi'(t) = d(y(t))\phi'(t)$. So $y'(t) \upuparrows d(y(t))$ because $\phi'(t) > 0$.

Let now $x'(t) \upuparrows d(x(t))$ that is that is there is a strictly positive function $c(t)$ such that $x'(t) = c(t)d(x(t))$.

If $x'(t)$ is zero everywhere, then $d(x(t)) = 0$ and thus $x'(t) = d(x(t))$ that is x is an Integral curve. Note that y is a reparameterization of itself.

We can assume that $x'(t) \neq 0$ everywhere. Then $F(x(t)) \neq 0$. We have that $c(t) = \frac{||x'(t)||}{||d(x(t))||}$ is a continuous function. FiXme: Does it work for non-normed spaces?

Let $y(u(t)) = x(t)$, where

$$u(t) = \int_0^t c(s)ds,$$

which is defined and finite because c is continuous and monotone (thus having inverse defined on its image) because c is positive.

Then

$$\begin{aligned} y'(u(t))u'(t) &= x'(t) \\ &= c(t)d(x(t)), \text{ so} \\ y'(u(t))c(t) &= c(t)d(y(u(t))) \\ y'(u(t)) &= d(y(u(t))) \end{aligned}$$

and letting $s = u(t)$ we have $y'(s) = d(y(s))$ for a reparameterization y of x. □

6.2. Vector space with additional coordinate. Consider the normed vector space with additional coordinate t.

Our vector field $d(t)$ induces vector field $\hat{d}(t, v) = (1, d(v))$ in this space. Also $\hat{f}(t) = (t, f(t))$.

PROPOSITION 2195. Let f be a D^1 function. f is an integral curve of d iff $\hat{f}$ is a reparametrized integral curve of $\hat{d}$.

PROOF. First note that $\hat{f}$ always has a nonzero derivative. $\hat{f}'(t) \upuparrows \hat{d}(\hat{f}(t)) \Leftrightarrow (1, f'(t)) \upuparrows (1, d(f(t))) \Leftrightarrow f'(t) = d(f(t))$. □

Thus we have reduced (for D^1 paths) being an integral curve to being a reparametrized integral curve. We will also describe being a reparametrized integral curve topologically (through funcoids).

6.3. Topological description of C^1 curves. Explicitly construct this funcoid as follows:

$R(d, \phi) = \left\{ \frac{v \in V}{\widehat{vd} < \phi, v \neq 0} \right\}$ for $d \neq 0$ and $R(0, \phi) = \{0\}$, where $\widehat{ab}$ is the angle between the vectors a and b, for a direction d and an angle ϕ.

DEFINITION 2196. $W(d) = \bigsqcap^{\mathsf{RLD}} \left\{ \frac{R(d,\phi)}{\phi \in \mathbb{R}, \phi > 0} \right\} \sqcap \bigsqcap^{\mathsf{RLD}}_{r>0} B_r(0)$. FiXme: This is defined for infinite dimensional case. FiXme: Consider also FCD instead of RLD.

PROPOSITION 2197. For finite dimensional case $\mathbb{R}^n$ we have $W(d) = \bigsqcap^{\mathsf{RLD}} \left\{ \frac{R(d,\phi)}{\phi \in \mathbb{R}, \phi > 0} \right\} \sqcap \Delta^{(\mathsf{RLD})n}$ where

$$\Delta^{(\mathsf{RLD})n} = \underbrace{\Delta \times^{\mathsf{RLD}} \cdots \times^{\mathsf{RLD}} \Delta}_{n \text{ times}}.$$

PROOF. ?? □

Finally our funcoids are the complete funcoids Q_+ and Q_- described by the formulas

$$\langle Q_+ \rangle^* @\{p\} = \langle p+ \rangle W(d(p)) \quad \text{and} \quad \langle Q_- \rangle^* @\{p\} = \langle p+ \rangle W(-d(p)).$$

Here Δ is taken from the "counter-examples" section.

In other words,

$$Q_+ = \bigsqcup_{p \in \mathbb{R}} (@\{p\} \times^{\mathsf{FCD}} \langle p+ \rangle W(d(p))); \quad Q_- = \bigsqcup_{p \in \mathbb{R}} (@\{p\} \times^{\mathsf{FCD}} \langle p+ \rangle W(-d(p))).$$

That is $\langle Q_+ \rangle^* @\{p\}$ and $\langle Q_- \rangle^* @\{p\}$ are something like infinitely small spherical sectors (with infinitely small aperture and infinitely small radius).

FiXme: Describe the co-complete funcoids reverse to these complete funcoids.

THEOREM 2198. A D^1 curve f is an reparametrized integral curve for a direction field d iff $f \in \mathrm{C}(\iota_D|\mathbb{R}|_>, Q_+) \cap \mathrm{C}(\iota_D|\mathbb{R}|_<, Q_-)$.

PROOF. Equivalently transform $f \in \mathrm{C}(\iota_D|\mathbb{R}|, Q_+)$; $f \circ \iota_D|\mathbb{R}| \sqsubseteq Q_+ \circ f$; $\langle f \circ \iota_D|\mathbb{R}| \rangle^* @\{t\} \sqsubseteq \langle Q_+ \circ f \rangle^* @\{t\}$; $\langle f \rangle^* \Delta_>(t) \sqcap D \sqsubseteq \langle Q_+ \rangle^* f(t)$; $\langle f \rangle^* \Delta_>(t) \sqsubseteq \langle Q_+ \rangle^* f(t)$; $\langle f \rangle^* \Delta_>(t) \sqsubseteq f(t) + W(D(f(t)))$; $\langle f \rangle^* \Delta_>(t) - f(t) \sqsubseteq W(D(f(t)))$;

$$\forall r > 0, \phi > 0 \exists \delta > 0 : \langle f \rangle^* (]t; t+\delta[) - f(t) \subseteq R(d(f(t)), \phi) \cap B_r(f(t));$$

$$\forall r > 0, \phi > 0 \exists \delta > 0 \forall 0 < \gamma < \delta : \langle f \rangle^* (]t; t+\gamma[) - f(t) \subseteq R(d(f(t)), \phi) \cap B_r(f(t));$$

$$\forall r > 0, \phi > 0 \exists \delta > 0 \forall 0 < \gamma < \delta : \frac{\langle f \rangle^* (]t; t+\gamma[) - f(t)}{\gamma} \subseteq R(d(f(t)), \phi) \cap B_{r/\delta}(f(t));$$

$$\forall r > 0, \phi > 0 \exists \delta > 0 : \partial_+ f(t) \subseteq R(d(f(t)), \phi) \cap B_{r/\delta}(f(t));$$

$$\forall r > 0, \phi > 0 : \partial_+ f(t) \subseteq R(d(f(t)), \phi);$$

$$\partial_+ f(t) \upuparrows d(f(t))$$

where ∂_+ is the right derivative.

In the same way we derive that $\mathrm{C}(|\mathbb{R}|_<, Q_-) \Leftrightarrow \partial_- f(t) \upuparrows d(f(t))$.

Thus $f'(t) \upuparrows d(f(t))$ iff $f \in \mathrm{C}(|\mathbb{R}|_>, Q_+) \cap \mathrm{C}(|\mathbb{R}|_<, Q_-)$. □

6.4. C^n curves. We consider the differential equation $f'(t) = d(f(t))$.

We can consider this equation in any topological vector space V (https://en.wikipedia.org/wiki/Frechet_derivative), see also https://math.stackexchange.com/q/2977274/4876. Note that I am not an expert in topological vector spaces and thus my naive generalizations may be wrong in details.

n-th derivative $f^{(n)}(t) = d_n(f(t))$; $f^{(n+1)}(t) = d'_n(f(t)) \circ f'(t) = d'_n(f(t)) \circ d(f(t))$. So $d_{n+1}(y) = d'_n(y) \circ d(y)$.

Given a point $y \in V$ define

$$R^n(y) = \left\{ \frac{v \in V}{\widehat{v d_0(y)} < \frac{d_1}{1!}(y)|v| + \frac{d_2(y)}{2!}|v|^2 + \cdots + \frac{d_{n-1}(y)}{(n-1)!}|v|^{n-1} + O(|v|^n), v \neq 0} \right\}$$

for $d_0(y) \neq 0$ and $R^n = \{0\}$ if $d_0(y) = 0$.

DEFINITION 2199. $R^\infty(y) = R^0(y) \sqcap R^1(y) \sqcap R^2(y) \sqcap \dots$.

FiXme: It does not work: https://math.stackexchange.com/a/2978532/4876.

DEFINITION 2200. $W^n(y) = R^n(y) \sqcap \prod^{\mathsf{RLD}}_{r>0} B_r(0)$; $W^\infty(y) = R^\infty(y) \sqcap \prod^{\mathsf{RLD}}_{r>0} B_r(0)$.

Finally our funcoids are the complete funcoids Q^n_+ and Q^n_- described by the formulas

$$\left\langle Q^n_+ \right\rangle^* @\{p\} = \langle p+\rangle W^n(p) \quad \text{and} \quad \left\langle Q^n_- \right\rangle^* @\{p\} = \langle p+\rangle W^{-n}(p)$$

where W^- is W for the reverse vector field $-d(y)$.

FiXme: Related questions: http://math.stackexchange.com/q/1884856/4876 http://math.stackexchange.com/q/107460/4876 http://mathoverflow.net/q/88501

LEMMA 2201. Let for every x in the domain of the path for small $t > 0$ we have $f(x+t) \in R^n(F(f(x)))$ and $f(x-t) \in R^n(-F(f(x)))$. Then f is C^n smooth.

PROOF. FiXme: Not yet proved!
See also http://math.stackexchange.com/q/1884930/4876. □

CONJECTURE 2202. A path f is conforming to the above differentiable equation and C^n (where n is natural or infinite) smooth iff $f \in \mathrm{C}(\iota_D|\mathbb{R}|_>, Q^n_+) \cap \mathrm{C}(\iota_D|\mathbb{R}|_<, Q^n_-)$.

PROOF. Reverse implication follows from the lemma.
Let now a path f is C^n. Then

$$f(x+t) = \sum_{i=0}^{n} f^{(i)}(x)\frac{t^i}{i!} + o(t^i) = f(x) + f'(x)t + \sum_{i=2}^{n} f^{(i)}(x)\frac{t^i}{i!} + o(t^i)$$

□

6.5. Plural funcoids. Take I_+ and Q_+ as described above in forward direction and I_- and Q_- in backward direction. Then

$$f \in \mathrm{C}(I_+, Q_+) \wedge f \in \mathrm{C}(I_-, Q_-) \Leftrightarrow f \times f \in \mathrm{C}(I_+ \times^{(A)} I_-, Q_+ \times^{(A)} Q_-)?$$

To describe the above we can introduce new term *plural funcoids*. This is simply a map from an index set to funcoids. Composition is defined componentwise. Order is defined as product order. Well, do we need this? Isn't it the same as infimum product of funcoids?

6.6. Multiple allowed directions per point.

$$\langle Q \rangle^* @\{p\} = \bigsqcup_{d \in d(p)} \langle p+\rangle W(d).$$

It seems (check!) that solutions not only of differential equations but also of difference equations can be expressed as paths in funcoids.

CHAPTER 4

Extending Galois connections between funcoids and reloids

Definition 2203.

1°. $\Phi_* f = \lambda b \in \mathfrak{B} : \bigsqcup \left\{ \frac{x \in \mathfrak{A}}{f x \sqsubseteq b} \right\}$;

2°. $\Phi^* f = \lambda b \in \mathfrak{A} : \bigsqcap \left\{ \frac{x \in \mathfrak{B}}{f x \sqsupseteq b} \right\}$.

Proposition 2204.

1°. If f has upper adjoint then $\Phi_* f$ is the upper adjoint of f.

2°. If f has lower adjoint then $\Phi^* f$ is the lower adjoint of f.

Proof. By theorem 131. □

Lemma 2205. $\Phi^* (\mathsf{RLD})_{\mathrm{out}} = (\mathsf{FCD})$.

Proof. $(\Phi^* (\mathsf{RLD})_{\mathrm{out}}) f = \bigsqcap \left\{ \frac{g \in \mathsf{FCD}}{(\mathsf{RLD})_{\mathrm{out}} g \sqsupseteq f} \right\} = \bigsqcap^{\mathsf{FCD}} \left\{ \frac{g \in \mathbf{Rel}}{(\mathsf{RLD})_{\mathrm{out}} g \sqsupseteq f} \right\} = \bigsqcap^{\mathsf{FCD}} \left\{ \frac{g \in \mathbf{Rel}}{g \sqsupseteq f} \right\} = (\mathsf{FCD}) f$. □

Lemma 2206. $\Phi_* (\mathsf{RLD})_{\mathrm{out}} \neq (\mathsf{FCD})$.

Proof. $(\Phi_* (\mathsf{RLD})_{\mathrm{out}}) f = \bigsqcup \left\{ \frac{g \in \mathsf{FCD}}{(\mathsf{RLD})_{\mathrm{out}} g \sqsubseteq f} \right\}$
$(\Phi_* (\mathsf{RLD})_{\mathrm{out}}) \bot \neq \bot$. □

Lemma 2207. $\Phi^* (\mathsf{FCD}) = (\mathsf{RLD})_{\mathrm{out}}$.

Proof. $(\Phi^* (\mathsf{FCD})) f = \bigsqcap \left\{ \frac{g \in \mathsf{RLD}}{(\mathsf{FCD}) g \sqsupseteq f} \right\} = \bigsqcap^{\mathsf{RLD}} \left\{ \frac{g \in \mathbf{Rel}}{(\mathsf{FCD}) g \sqsupseteq f} \right\} = \bigsqcap^{\mathsf{RLD}} \left\{ \frac{g \in \mathbf{Rel}}{g \sqsupseteq f} \right\} = (\mathsf{RLD})_{\mathrm{out}} f$. □

Lemma 2208. $\Phi_* (\mathsf{RLD})_{\mathrm{in}} = (\mathsf{FCD})$.

Proof. $(\Phi_* (\mathsf{RLD})_{\mathrm{in}}) f = \bigsqcup \left\{ \frac{g \in \mathsf{FCD}}{(\mathsf{RLD})_{\mathrm{in}} g \sqsubseteq f} \right\} = \bigsqcup \left\{ \frac{g \in \mathsf{FCD}}{g \sqsubseteq (\mathsf{FCD}) f} \right\} = (\mathsf{FCD}) f$. □

Theorem 2209. The picture at figure 1 describes values of functions Φ_* and Φ^*. All nodes of this diagram are distinct.

Proof. Follows from the above lemmas. □

Figure 1

$$\begin{tikzcd}
(\mathsf{FCD}) \arrow[r, leftrightarrow, "\Phi_*"] \arrow[d, leftrightarrow, "\Phi^*"'] & (\mathsf{RLD})_{\mathrm{in}} \arrow[l, shift left=2, "\Phi^*"] \\
(\mathsf{RLD})_{\mathrm{out}} \arrow[d, "\Phi_*"] & \\
\text{other} &
\end{tikzcd}$$

QUESTION 2210. What is at the node "other"?

Trying to answer this question:

LEMMA 2211. $(\Phi_*(\mathsf{RLD})_{\mathrm{out}})\bot = \Omega^{\mathsf{FCD}}$.

PROOF. We have $(\mathsf{RLD})_{\mathrm{out}}\Omega^{\mathsf{FCD}} = \bot$. $x \not\sqsubseteq \Omega^{\mathsf{FCD}} \Rightarrow (\mathsf{RLD})_{\mathrm{out}}x \sqsupseteq \operatorname{Cor} x \sqsupset \bot$. Thus $\max\left\{ \frac{x\in\mathsf{FCD}}{(\mathsf{RLD})_{\mathrm{out}}x=\bot} \right\} = \Omega^{\mathsf{FCD}}$.

So $(\Phi_*(\mathsf{RLD})_{\mathrm{out}})\bot = \Omega^{\mathsf{FCD}}$. □

CONJECTURE 2212. $(\Phi_*(\mathsf{RLD})_{\mathrm{out}})f = \Omega^{\mathsf{FCD}} \sqcup (\mathsf{FCD})f$.

The above conjecture looks not natural, but I do not see a better alternative formula.

QUESTION 2213. What happens if we keep applying Φ^* and Φ_* to the node "other"? Will we this way get a finite or infinite set?

CHAPTER 5

Boolean funcoids

1. One-element boolean lattice

Let $\mathfrak{A}$ be a boolean lattice and $\mathfrak{B} = \mathscr{P}0$. It's sole element is $\bot$.

$f \in \mathsf{pFCD}(\mathfrak{A}; \mathfrak{B}) \Leftrightarrow \forall X \in \mathfrak{A} : (\langle f \rangle X \not\asymp \bot \Leftrightarrow \langle f^{-1} \rangle \bot \not\asymp X) \Leftrightarrow \forall X \in \mathfrak{A} : (0 \Leftrightarrow \langle f^{-1} \rangle \bot \not\asymp X) \Leftrightarrow \forall X \in \mathfrak{A} : \langle f^{-1} \rangle \bot \asymp X \Leftrightarrow \forall X \in \mathfrak{A} : \langle f^{-1} \rangle \bot = \bot^{\mathfrak{A}} \Leftrightarrow \langle f^{-1} \rangle \bot = \bot^{\mathfrak{A}} \Leftrightarrow \langle f^{-1} \rangle = \{(\bot; \bot^{\mathfrak{A}})\}$.

Thus $\operatorname{card} \mathsf{pFCD}(\mathfrak{A}; \mathscr{P}0) = 1$.

2. Two-element boolean lattice

Consider the two-element boolean lattice $\mathfrak{B} = \mathscr{P}1$.

Let f be a pointfree protofuncoid from $\mathfrak{A}$ to $\mathfrak{B}$ (that is $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)$ where $\alpha \in \mathfrak{B}^{\mathfrak{A}}$, $\beta \in \mathfrak{A}^{\mathfrak{B}}$).

$f \in \mathsf{pFCD}(\mathfrak{A}; \mathfrak{B}) \Leftrightarrow \forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\langle f \rangle X \not\asymp Y \Leftrightarrow \langle f^{-1} \rangle Y \not\asymp X) \Leftrightarrow \forall X \in \mathfrak{A}, Y \in \mathfrak{B} : ((0 \in \langle f \rangle X \wedge 0 \in Y) \vee (1 \in \langle f \rangle X \wedge 1 \in Y) \Leftrightarrow \langle f^{-1} \rangle Y \not\asymp X)$.

$T = \left\{ \frac{X \in \mathfrak{A}}{0 \in \langle f \rangle X} \right\}$ is an ideal. Really: That it's an upper set is obvious. Let $P \cup Q \in \left\{ \frac{X \in \mathfrak{A}}{0 \in \langle f \rangle X} \right\}$. Then $0 \in \langle f \rangle (P \cup Q) = \langle f \rangle P \cup \langle f \rangle Q$; $0 \in \langle f \rangle P \vee 0 \in \langle f \rangle Q$.

Similarly $S = \left\{ \frac{X \in \mathfrak{A}}{1 \in \langle f \rangle X} \right\}$ is an ideal.

Let now $T, S \in \mathscr{P}\mathfrak{A}$ be ideals. Can we restore $\langle f \rangle$? Yes, because we know $0 \in \langle f \rangle X$ and $1 \in \langle f \rangle X$ for every $X \in \mathfrak{A}$.

So it is equivalent to $\forall X \in \mathfrak{A}, Y \in \mathfrak{B} : ((X \in T \wedge 0 \in Y) \vee (X \in S \wedge 1 \in Y) \Leftrightarrow \langle f^{-1} \rangle Y \not\asymp X)$.

$f \in \mathsf{pFCD}(\mathfrak{A}; \mathfrak{B})$ is equivalent to conjunction of all rows of this table:

Y	equality
$\emptyset$	$\langle f^{-1} \rangle \emptyset = \emptyset$
$\{0\}$	$X \in T \Leftrightarrow \langle f^{-1} \rangle \{0\} \not\asymp X$
$\{1\}$	$X \in S \Leftrightarrow \langle f^{-1} \rangle \{1\} \not\asymp X$
$\{0,1\}$	$X \in T \vee X \in S \Leftrightarrow \langle f^{-1} \rangle \{0, 1\} \not\asymp X$

Simplified:

Y	equality
$\emptyset$	$\langle f^{-1} \rangle \emptyset = \emptyset$
$\{0\}$	$T = \partial \langle f^{-1} \rangle \{0\}$
$\{1\}$	$S = \partial \langle f^{-1} \rangle \{1\}$
$\{0,1\}$	$T \cup S = \partial \langle f^{-1} \rangle \{0, 1\}$

From the last table it follows that T and S are principal ideals.

So we can take arbitrary either $\langle f^{-1} \rangle \{0\}$, $\langle f^{-1} \rangle \{1\}$ or principal ideals T and S.

In other words, we take $\langle f^{-1} \rangle \{0\}$, $\langle f^{-1} \rangle \{1\}$ arbitrary and independently. So we have $\mathsf{pFCD}(\mathfrak{A}; \mathfrak{B})$ equivalent to product of two instances of $\mathfrak{A}$. So it a boolean lattice. FiXme: I messed product with disjoint union below.)

3. Finite boolean lattices

We can assume $\mathfrak{B} = \mathscr{P}B$ for a set B, $\operatorname{card} B = n$. Then

$f \in \mathsf{pFCD}(\mathfrak{A}; \mathfrak{B}) \Leftrightarrow \forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\langle f \rangle X \not\asymp Y \Leftrightarrow \langle f^{-1} \rangle Y \not\asymp X) \Leftrightarrow \forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\exists i \in Y : i \in \langle f \rangle X \Leftrightarrow \langle f^{-1} \rangle Y \not\asymp X)$.

Having values of $\langle f^{-1} \rangle \{i\}$ we can restore all $\langle f^{-1} \rangle Y$. [need this paragraph?]

Let $T_i = \left\{ \frac{X \in \mathfrak{A}}{i \in \langle f \rangle X} \right\}$.

Let now $T_i \in \mathscr{P}\mathfrak{A}$ be ideals. Can we restore $\langle f \rangle$? Yes, because we know $i \in \langle f \rangle X$ for every $X \in \mathfrak{A}$.

So, it is equivalent to:

$$\forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\exists i \in Y : X \in T_i \Leftrightarrow \langle f^{-1} \rangle Y \not\asymp X). \tag{1}$$

Lemma 2214. The formula (1) is equivalent to:

$$\forall X \in \mathfrak{A}, i \in B : (X \in T_i \Leftrightarrow \langle f^{-1} \rangle \{i\} \not\asymp X). \tag{2}$$

Proof. (1)⇒(2). Just take $Y = \{i\}$.

(2)⇒(1). Let (2) holds. Let also $X \in \mathfrak{A}, Y \in \mathfrak{B}$. Then $\langle f^{-1} \rangle Y \not\asymp X \Leftrightarrow \bigcup_{i \in Y} \langle f^{-1} \rangle \{i\} \not\asymp X \Leftrightarrow \exists i \in Y : \langle f^{-1} \rangle \{i\} \not\asymp X \Leftrightarrow \exists i \in Y : X \in T_i$.

□

Further transforming: $\forall i \in B : T_i = \partial \langle f^{-1} \rangle \{i\}$.

So $\langle f^{-1} \rangle \{i\}$ are arbitary elements of $\mathfrak{B}$ and T_i are corresponding arbitrary principal ideals.

In other words, $\mathsf{pFCD}(\mathfrak{A}; \mathfrak{B}) \cong \mathfrak{A} \Pi \dots \Pi \mathfrak{A}$ ($\operatorname{card} B$ times). Thus $\mathsf{pFCD}(\mathfrak{A}; \mathfrak{B})$ is a boolean lattice.

4. About infinite case

Let $\mathfrak{A}$ be a complete boolean lattice, $\mathfrak{B}$ be an atomistic boolean lattice.

$f \in \mathsf{pFCD}(\mathfrak{A}; \mathfrak{B}) \Leftrightarrow \forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\langle f \rangle X \not\asymp Y \Leftrightarrow \langle f^{-1} \rangle Y \not\asymp X) \Leftrightarrow \forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\exists i \in \operatorname{atoms} Y : i \in \operatorname{atoms} \langle f \rangle X \Leftrightarrow \langle f^{-1} \rangle Y \not\asymp X)$.

Let $T_i = \left\{ \frac{X \in \mathfrak{A}}{i \in \operatorname{atoms} \langle f \rangle X} \right\}$.

T_i is an ideal: Really: That it's an upper set is obvious. Let $P \cup Q \in \left\{ \frac{X \in \mathfrak{A}}{i \in \operatorname{atoms} \langle f \rangle X} \right\}$. Then $i \in \operatorname{atoms} \langle f \rangle (P \cup Q) = \operatorname{atoms} \langle f \rangle P \cup \operatorname{atoms} \langle f \rangle Q$; $i \in \langle f \rangle P \vee i \in \langle f \rangle Q$.

Let now $T_i \in \mathscr{P}\mathfrak{A}$ be ideals. Can we restore $\langle f \rangle$? Yes, because we know $i \in \operatorname{atoms} \langle f \rangle X$ for every $X \in \mathfrak{A}$ and $\mathfrak{B}$ is atomistic.

So, it is equivalent to:

$$\forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\exists i \in \operatorname{atoms} Y : X \in T_i \Leftrightarrow \langle f^{-1} \rangle Y \not\asymp X). \tag{3}$$

Lemma 2215. The formula (3) is equivalent to:

$$\forall X \in \mathfrak{A}, i \in \operatorname{atoms}^{\mathfrak{B}} : (X \in T_i \Leftrightarrow \langle f^{-1} \rangle i \not\asymp X). \tag{4}$$

Proof. (3)⇒(4). Let (3) holds. Take $Y = i$. Then $\operatorname{atoms} Y = \{i\}$ and thus $X \in T_i \Leftrightarrow \exists i \in \operatorname{atoms} Y : X \in T_i \Leftrightarrow \langle f^{-1} \rangle Y \not\asymp X \Leftrightarrow \langle f^{-1} \rangle i \not\asymp X$.

(4)⇒(3). Let (2) holds. Let also $X \in \mathfrak{A}$, $Y \in \mathfrak{B}$. Then $\langle f^{-1} \rangle Y \not\asymp X \Leftrightarrow \langle f^{-1} \rangle \bigsqcup \operatorname{atoms} Y \not\asymp X \Leftrightarrow \bigsqcup_{i \in \operatorname{atoms} Y} \langle f^{-1} \rangle i \not\asymp X \Leftrightarrow \exists i \in \operatorname{atoms} Y : \langle f^{-1} \rangle i \not\asymp X \Leftrightarrow \exists i \in \operatorname{atoms} Y : X \in T_i$.

□

Further equivalently transforming: $\forall i \in \operatorname{atoms}^{\mathfrak{B}} : T_i = \partial \langle f^{-1} \rangle i$.

So $\langle f^{-1} \rangle i$ are arbitary elements of $\mathfrak{B}$ and T_i are corresponding arbitrary principal ideals.

In other words, $\mathsf{pFCD}(\mathfrak{A}; \mathfrak{B}) \cong \prod_{i \in \operatorname{card} \operatorname{atoms}^{\mathfrak{B}}} \mathfrak{A}$. Thus $\mathsf{pFCD}(\mathfrak{A}; \mathfrak{B})$ is a boolean lattice.

So finally we have a very weird theorem, which is a partial solution for the above open problem (The weirdness is in its partiality and asymmetry):

THEOREM 2216. If $\mathfrak{A}$ is a complete boolean lattice and $\mathfrak{B}$ is an atomistic boolean lattice (or vice versa), then $\mathsf{pFCD}(\mathfrak{A}; \mathfrak{B})$ is a boolean lattice.

[4] proves "THEOREM 4.6. Let A, B be bounded posets. $A \otimes B$ is a completely distributive complete Boolean lattice iff A and B are completely distributive Boolean lattices." (where $A \otimes B$ is equivalent to the set of Galois connections between A and B) and other interesting results.

CHAPTER 6

Interior funcoids

Having a funcoid f let define *interior funcoid* f°.

DEFINITION 2217. Let $f \in \mathsf{FCD}(A, B) = \mathsf{pFCD}(\mathscr{T}A, \mathscr{T}B)$ be a co-complete funcoid. Then $f^\circ \in \mathsf{pFCD}(\operatorname{dual}\mathscr{T}A, \operatorname{dual}\mathscr{T}B)$ is defined by the formula $\langle f^\circ\rangle^* X = \overline{\langle f\rangle\overline{X}}$.

PROPOSITION 2218. Pointfree funcoid f° exists and is unique.

PROOF. $X \mapsto \overline{\langle f\rangle\overline{X}}$ is a component of pointfree funcoid $\operatorname{dual}\mathscr{T}A \to \operatorname{dual}\mathscr{T}B$ iff $\langle f\rangle$ is a component of the corresponding pointfree funcoid $\mathscr{T}A \to \mathscr{T}B$ that is essentially component of the corresponding funcoid $\mathsf{FCD}(A, B)$ what holds for a unique funcoid. □

It can be also defined for arbitrary funcoids by the formula $f^\circ = (\operatorname{CoCompl} f)^\circ$.

OBVIOUS 2219. f° is co-complete.

THEOREM 2220. The following values are pairwise equal for a co-complete funcoid f and $X \in \mathscr{T}\operatorname{Src} f$:

1°. $\langle f^\circ\rangle^* X$;
2°. $\left\{ \frac{y \in \operatorname{Dst} f}{\langle f^{-1}\rangle^*\{y\} \sqsubseteq X} \right\}$
3°. $\bigsqcup\left\{ \frac{Y \in \mathscr{T}\operatorname{Dst} f}{\langle f^{-1}\rangle^* Y \sqsubseteq X} \right\}$
4°. $\bigsqcup\left\{ \frac{\mathcal{Y} \in \mathscr{F}\operatorname{Dst} f}{\langle f^{-1}\rangle\mathcal{Y} \sqsubseteq X} \right\}$

PROOF.

1°⇒2°. $\left\{ \frac{y \in \operatorname{Dst} f}{\langle f^{-1}\rangle^*\{y\} \sqsubseteq X} \right\} = \left\{ \frac{x \in \operatorname{Dst} f}{\langle f^{-1}\rangle^*\{x\} \asymp \overline{X}} \right\} = \left\{ \frac{x \in \operatorname{Dst} f}{\{x\} \asymp \langle f\rangle\overline{X}} \right\} = \overline{\langle f\rangle\overline{X}} = \langle f^\circ\rangle^* X$.

2°⇒3°. If $\langle f^{-1}\rangle^* Y \sqsubseteq X$ then (by completeness of f^{-1}) $Y = \left\{ \frac{y \in Y}{\langle f^{-1}\rangle^*\{y\} \sqsubseteq X} \right\}$ and thus
$$\bigsqcup\left\{ \frac{Y \in \mathscr{T}\operatorname{Dst} f}{\langle f^{-1}\rangle^* Y \sqsubseteq X} \right\} \sqsubseteq \left\{ \frac{y \in \operatorname{Dst} f}{\langle f^{-1}\rangle^*\{y\} \sqsubseteq X} \right\}.$$
The reverse inequality is obvious.

3°⇒4°. It's enough to prove that if $\langle f^{-1}\rangle\mathcal{Y} \sqsubseteq X$ for $\mathcal{Y} \in \mathscr{F}\operatorname{Dst} f$ then exists $Y \in \operatorname{up}\mathcal{Y}$ such that $\langle f^{-1}\rangle^* Y \sqsubseteq X$. Really let $\langle f^{-1}\rangle\mathcal{Y} \sqsubseteq X$. Then $\bigsqcap\langle\langle f^{-1}\rangle^*\rangle^* \operatorname{up}\mathcal{Y} \sqsubseteq X$ and thus exists $Y \in \operatorname{up}\mathcal{Y}$ such that $\langle f^{-1}\rangle^* Y \sqsubseteq X$ by properties of generalized filter bases.

□

This coincides with the customary definition of interior in topological spaces.

PROPOSITION 2221. $f^{\circ\circ} = f$ for every funcoid f.

PROOF. $\langle f^{\circ\circ}\rangle^* X = \neg\neg\langle f\rangle\neg\neg X = \langle f\rangle X$. □

PROPOSITION 2222. Let $g \in \mathsf{FCD}(A, B)$, $f \in \mathsf{FCD}(B, C)$, $h \in \mathsf{FCD}(A, C)$ for some sets A. B, C.

$g \sqsubseteq f^\circ \circ h \Leftrightarrow f^{-1} \circ g \sqsubseteq h$, provided f and h are co-complete.

PROOF. $g \sqsubseteq f^{\circ} \circ h \Leftrightarrow \forall X \in A : \langle g \rangle^* X \sqsubseteq \langle f^{\circ} \circ h \rangle^* X \Leftrightarrow \forall X \in A : \langle g \rangle^* X \sqsubseteq \langle f^{\circ} \rangle^* \langle h \rangle^* X \Leftrightarrow \forall X \in A : \langle g \rangle^* X \sqsubseteq \neg \langle f \rangle^* \neg \langle h \rangle^* X \Leftrightarrow \forall X \in A : \langle g \rangle^* X \asymp \langle f \rangle^* \neg \langle h \rangle^* X \Leftrightarrow \forall X \in A : \left\langle f^{-1} \right\rangle^* \langle g \rangle^* X \asymp \neg \langle h \rangle^* X \Leftrightarrow \forall X \in A : \left\langle f^{-1} \right\rangle^* \langle g \rangle^* X \sqsubseteq \langle h \rangle^* X \Leftrightarrow \forall X \in A : \left\langle f^{-1} \circ g \right\rangle^* X \sqsubseteq \langle h \rangle^* X \Leftrightarrow f^{-1} \circ g \sqsubseteq h.$ □

REMARK 2223. The above theorem allows to get rid of interior funcoids (and use only “regular” funcoids) in some formulas.

CHAPTER 7

Filterization of pointfree funcoids

Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. By corollary 515 we have that $\mathfrak{A}$ and $\mathfrak{B}$ are complete lattices.

Let f be a pointfree funcoid $\mathfrak{Z}_0 \to \mathfrak{Z}_1$. Define pointfree funcoid $\uparrow f$ (*filterization* of f) by the formulas

$$\langle \uparrow f \rangle \mathcal{X} = \bigsqcap^{\mathfrak{B}}_{X \in \operatorname{up} \mathcal{X}} \langle f \rangle X \quad \text{and} \quad \left\langle \uparrow f^{-1} \right\rangle \mathcal{Y} = \bigsqcap^{\mathfrak{A}}_{Y \in \operatorname{up} \mathcal{Y}} \left\langle f^{-1} \right\rangle Y.$$

Proposition 2224. $\uparrow f$ is a pointfree funcoid.

Proof.

$$\begin{aligned}
\mathcal{Y} \not\asymp \langle \uparrow f \rangle \mathcal{X} \Leftrightarrow \mathcal{Y} \not\asymp \bigsqcap^{\mathfrak{B}}_{X \in \operatorname{up} \mathcal{X}} \langle f \rangle X \Leftrightarrow \\
\bigsqcap^{\mathfrak{B}}_{X \in \operatorname{up} \mathcal{X}} (\mathcal{Y} \sqcap^{\mathfrak{B}} \langle f \rangle X) \neq \bot \Leftrightarrow \text{ (corollary 570*)} \\
\forall X \in \operatorname{up} \mathcal{X} : \mathcal{Y} \sqcap^{\mathfrak{B}} \langle f \rangle X \neq \bot \Leftrightarrow \text{ (theorem 534)} \\
\forall X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y} : Y \sqcap^{\mathfrak{B}} \langle f \rangle X \neq \bot \Leftrightarrow \text{ (corollary 533)} \\
\forall X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y} : Y \sqcap^{\mathfrak{Z}_1} \langle f \rangle X \neq \bot \Leftrightarrow \\
\forall X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y} : X \; [f] \; Y.
\end{aligned}$$

* To apply corollary 570 we need to show that $\left\{ \frac{\mathcal{Y} \sqcap^{\mathfrak{B}} \langle f \rangle X}{X \in \operatorname{up} \mathcal{X}} \right\}$ is a generalized filter base. To show it is enough to show that $\left\{ \frac{\langle f \rangle X}{X \in \operatorname{up} \mathcal{X}} \right\}$ is a generalized filter base. But this easily follows from proposition 1598 and 576.

Similarly $\mathcal{X} \not\asymp \left\langle \uparrow f^{-1} \right\rangle \mathcal{Y} \Leftrightarrow \forall X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y} : X \; [f] \; Y$. Thus $\mathcal{Y} \not\asymp \langle \uparrow f \rangle \mathcal{X} \Leftrightarrow \mathcal{X} \not\asymp \left\langle \uparrow f^{-1} \right\rangle \mathcal{Y}$. □

Proposition 2225. The above defined $\uparrow$ is an injection.

Proof. $\langle \uparrow f \rangle X = \bigsqcap^{\mathfrak{B}}_{X' \in \operatorname{up} X} \langle f \rangle X' = \min_{X' \in \operatorname{up} X} \langle f \rangle X' = \langle f \rangle X$. So $\langle f \rangle$ is determined by $\langle \uparrow f \rangle$. Likewise $\left\langle f^{-1} \right\rangle$ is determined by $\left\langle \uparrow f^{-1} \right\rangle$. □

Conjecture 2226. (Non generalizing of theorem 1707) Pointfree funcoids f between some: a. atomistic but non-complete; b. complete but non-atomistic boolean lattices $\mathfrak{Z}_0$ and $\mathfrak{Z}_1$ do not bijectively correspond to morphisms $p \in \mathbf{Rel}(\operatorname{atoms} \mathfrak{Z}_0, \operatorname{atoms} \mathfrak{Z}_1)$ by the formulas:

$$\langle f \rangle X = \bigsqcup \langle p \rangle^* \operatorname{atoms} X, \quad \left\langle f^{-1} \right\rangle Y = \bigsqcup \left\langle p^{-1} \right\rangle^* \operatorname{atoms} Y;$$

$$(x, y) \in \operatorname{GR} p \Leftrightarrow y \in \operatorname{atoms} \langle f \rangle x \Leftrightarrow x \in \operatorname{atoms} \left\langle f^{-1} \right\rangle y.$$

CHAPTER 8

Systems of sides

Now we will consider a common generalization of (some of pointfree) funcoids and (some of) Galois connections. The main purpose of this is general theorem 2274 below.

First consider some properties of Galois connections:

1. More on Galois connections

Here I will denote $\langle f\rangle$ the lower adjoint of a Galois connection f. FiXme: Switch to this notation in the book?

Let GAL be the category of Galois connections. FiXme: Need to decide whether use $\mathsf{GAL}(A, B)$ or $A \otimes B$.

I will denote $(f, g)^{-1} = (g, f)$ for a Galois connection (f, g).

We will order Galois connections by the formula

$$f \sqsubseteq g \Leftrightarrow \langle f\rangle \sqsubseteq \langle g\rangle \Leftrightarrow \langle f^{-1}\rangle \sqsupseteq \langle g^{-1}\rangle.$$

OBVIOUS 2227. This defines a partial order on the set of Galois connections between any two (fixed) posets.

PROPOSITION 2228. If f and g are Galois connections (between a join-semilattice $\mathfrak{A}$ and a meet-semilattice $\mathfrak{B}$), then there exists a Galois connection $f \sqcup g$ determined by the formula $\langle f \sqcup g\rangle x = \langle f\rangle x \sqcup \langle g\rangle x$.

PROOF. It is enough to prove that

$$(x \mapsto \langle f\rangle x \sqcup \langle g\rangle x, y \mapsto \langle f^{-1}\rangle y \sqcap \langle g^{-1}\rangle y)$$

is a Galois connection that is that

$$\langle f\rangle x \sqcup \langle g\rangle x \sqsubseteq y \Leftrightarrow x \sqsubseteq \langle f^{-1}\rangle y \sqcap \langle g^{-1}\rangle y$$

for all relevant x and y.

Really,

$$\langle f\rangle x \sqcup \langle g\rangle x \sqsubseteq y \Leftrightarrow \langle f\rangle x \sqsubseteq y \wedge \langle g\rangle x \sqsubseteq y \Leftrightarrow \\ x \sqsubseteq \langle f^{-1}\rangle y \wedge x \sqsubseteq \langle g^{-1}\rangle y \Leftrightarrow x \sqsubseteq \langle f^{-1}\rangle y \sqcap \langle g^{-1}\rangle y.$$

□

FiXme: Describe infinite join of Galois connections.

PROPOSITION 2229. If $\mathfrak{A}$ is a poset with least element, then $\langle a\rangle\bot = \bot$.

PROOF. $\langle a\rangle\bot \sqsubseteq y \Leftrightarrow \bot \sqsubseteq \langle a^{-1}\rangle y \Leftrightarrow 1$. Thus $\langle a\rangle\bot$ is the least element. □

PROPOSITION 2230. $(\mathfrak{A} \times \{\bot^{\mathfrak{B}}\}, \mathfrak{B} \times \{\top^{\mathfrak{A}}\})$ is the least Galois connection from a poset $\mathfrak{A}$ with greatest element to a poset $\mathfrak{B}$ with least element.

PROOF. Let's prove that it is a Galois connection. We need to prove

$$(\mathfrak{A} \times \{\bot^{\mathfrak{B}}\})x \sqsubseteq y \Leftrightarrow x \sqsubseteq (\mathfrak{B} \times \{\top^{\mathfrak{A}}\})y.$$

But this is trivially equivalent to $1 \Leftrightarrow 1$. Thus it's a Galois connection.

That it the least is obvious. □

COROLLARY 2231. $\langle\bot\rangle x = \bot$ for Galois connections from a poset $\mathfrak{A}$ with greatest element to a poset $\mathfrak{B}$ with least element. FiXme: Clarify.

THEOREM 2232. If $\mathfrak{A}$ and $\mathfrak{B}$ are bounded posets, then $\mathsf{GAL}(\mathfrak{A},\mathfrak{B})$ is bounded.

PROOF. That $\mathsf{GAL}(\mathfrak{A},\mathfrak{B})$ has least element was proved above. I will demonstrate that (α,β) is the greatest element of $\mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$ for

$$\alpha X = \begin{cases} \bot^{\mathfrak{B}} & \text{if } X = \bot^{\mathfrak{A}} \\ \top^{\mathfrak{B}} & \text{if } X \neq \bot^{\mathfrak{A}} \end{cases}; \quad \beta Y = \begin{cases} \top^{\mathfrak{A}} & \text{if } Y = \top^{\mathfrak{B}} \\ \bot^{\mathfrak{A}} & \text{if } Y \neq \top^{\mathfrak{B}} \end{cases}.$$

First prove $Y \sqsubseteq \alpha X \Leftrightarrow X \sqsubseteq \beta Y$.

Really $\alpha X \sqsubseteq Y \Leftrightarrow X = \bot^{\mathfrak{A}} \vee Y = \top^{\mathfrak{B}} \Leftrightarrow X \sqsubseteq \beta Y$.

That it is the greatest Galois connection between $\mathfrak{A}$ and $\mathfrak{B}$ easily follows from proposition 2229. □

THEOREM 2233. For every brouwerian lattice $x \mapsto c \sqcap x$ is a lower adjoint.

PROOF. By dual of theorem 154. □

EXERCISE 2234. Describe the corresponding upper adjoint, especially for the special case of boolean lattices.

2. Definition

DEFINITION 2235. *System of presides* is a functor $\Upsilon = (f \mapsto \langle f\rangle)$ from an ordered category to the category of functions between (small) bounded lattices, such that (for all relevant variables):

1°. Every Hom-set of $\operatorname{Src}\Upsilon$ is a bounded join-semilattice.
2°. $\langle a\rangle\bot = \bot$.
3°. $\langle a \sqcup b\rangle X = \langle a\rangle X \sqcup \langle b\rangle X$ (equivalent to Υ to be a join-semilattice homomorphism, if we order functions between small bounded lattices componentwise).

I call morphisms of such categories *sides*.[1]

REMARK 2236. We could generalize to functions between small join-semilattices with least elements instead of bounded lattices only, but this is not really necessary.

DEFINITION 2237. I will call objects of the source category of this functor simply *objects of the presides*.

DEFINITION 2238. *Bounded* system of presides is system of presides from an ordered category with bounded Hom-sets such that $X, Y \in \operatorname{Ob}\operatorname{Src}\Upsilon$ the following additional axioms hold for all suitable a:

1°. $\left\langle\bot^{\operatorname{Hom}(X,Y)}\right\rangle a = \bot$.
2°. $\left\langle\top^{\operatorname{Hom}(X,Y)}\right\rangle a = \top$ unless $a = \bot$

DEFINITION 2239. *System of presides with identities* is a system of presides with a morphism $\mathrm{id}_a \in \operatorname{Src}\Upsilon$ for every object $\mathfrak{A}$ of $\operatorname{Src}\Upsilon$ and $a \in \mathfrak{A}$ and the following additional axioms:

1°. $\mathrm{id}_c \sqsubseteq 1_{\mathfrak{A}}$ for every $c \in \mathfrak{A}$ where $\mathfrak{A}$ is an object of $\operatorname{Src}\Upsilon$.
2°. $\langle \mathrm{id}_c\rangle = (\lambda x \in \mathfrak{A} : x \sqcap c)$ for every $c \in \mathfrak{A}$ where $\mathfrak{A}$ is an object of $\operatorname{Src}\Upsilon$

DEFINITION 2240. *System of sides* is a system of presides which is both bounded and with identities.

[1] The idea for the name is that we consider one "side" $\langle f\rangle$ of a funcoid instead of both sides $\langle f\rangle$ and $\left\langle f^{-1}\right\rangle$.

PROPOSITION 2241. $\langle 1_{\mathfrak{A}}^{\operatorname{Src}\Upsilon}\rangle a = a$ for every system of presides.

PROOF. By properties of functors. □

DEFINITION 2242. I call a system of *monotone* presides a system of presides with additional axiom:

1°. $\langle a\rangle$ is monotone.

DEFINITION 2243. I call a system of *distributive* presides a system of presides with additional axiom:

1°. $\langle a\rangle(X \sqcup Y) = \langle a\rangle X \sqcup \langle a\rangle Y$.

OBVIOUS 2244. Every distributive system of presides is monotone.

PROPOSITION 2245. $\langle a \sqcap b\rangle X \sqsubseteq \langle a\rangle X \sqcap \langle b\rangle X$ for monotone systems of sides if Hom-sets are lattices.

DEFINITION 2246. A system of presides *with correct identities* is a system of presides with identities with additional axiom:

1°. $\mathrm{id}_b \circ \mathrm{id}_a = \mathrm{id}_{a\sqcap b}$.

PROPOSITION 2247. Every faithful system of presides with identities is with correct identities.

PROOF. $\langle \mathrm{id}_b \circ \mathrm{id}_a\rangle x = (\langle \mathrm{id}_b\rangle \circ \langle \mathrm{id}_a\rangle)x = \langle \mathrm{id}_b\rangle\langle \mathrm{id}_a\rangle x = b \sqcap a \sqcap x = \langle \mathrm{id}_{b\sqcap a}\rangle x$. Thus by faithfulness $\mathrm{id}_b \circ \mathrm{id}_a = \mathrm{id}_{b\sqcap a} = \mathrm{id}_{a\sqcap b}$. □

DEFINITION 2248. *Restricting* a side f to an object X is defined by the formula $f|_X = f \circ \mathrm{id}_X$.

DEFINITION 2249. *Image* of a preside is defined by the formula $\operatorname{im} f = \langle f\rangle\top$.

DEFINITION 2250. Protofuncoids *over* a set X of functors is a protofuncoid f such that $\langle f\rangle \in X \wedge \langle f^{-1}\rangle \in X$.

3. Concrete examples of sides

OBVIOUS 2251. The category **Rel** with $\langle f\rangle = \langle f\rangle^*$ for $f \in \mathbf{Rel}$ and usual id_c defines a distributive system of sides with correct identities.

3.1. Some subsides.

DEFINITION 2252. *Full subsystem* of a system Υ of presides is the functor Υ restricted to a full subcategory of $\operatorname{Src}\Upsilon$.

OBVIOUS 2253. Full subsystem of a system of presides is always a system of presides.

OBVIOUS 2254. Full subsystem of a bounded system of presides is always a bounded subsystem of presides.

OBVIOUS 2255.

1°. Full subsystem of a system of presides with identities is always with identities.
2°. Full subsystem of a system of presides with correct identities is always with correct identities.

OBVIOUS 2256. Full subsystem of a distributive system of presides is always a distributive system of presides.

OBVIOUS 2257. Full subsystem of a system of sides is always a system of sides.

3.2. Funcoids and pointfree funcoids.

PROPOSITION 2258. The category of pointfree funcoids between starrish join-semilattices with usual $\langle f \rangle$ defines a system of presides.

PROOF. Theorem 1627. □

PROPOSITION 2259. The category of pointfree funcoids between bounded starrish join-semilattices with usual $\langle f \rangle$ defines a system of bounded presides.

PROOF. Take the proof of theorem 1624 into account. □

PROPOSITION 2260. The category of pointfree funcoids from a starrish join-semilattices to a separable starrish join-semilattices defines a distributive system of presides.

PROOF. Theorem 1599. □

PROPOSITION 2261. The category of pointfree funcoids between starrish lattices with usual $\langle f \rangle$ and usual id_c defines a system of presides with correct identities.

PROOF. That it is with identities is obvious.
That it is with correct identities is obvious. □

OBVIOUS 2262. The category of pointfree funcoids between bounded starrish lattices with usual $\langle f \rangle$ and usual id_c defines a system of sides with correct identities.

PROPOSITION 2263. The category of funcoids with usual $\langle f \rangle$ and usual id_c defines a system of sides with correct identities.

PROOF. Because it can be considered a full subsystem of the category of pointfree funcoids between bounded starrish lattices with usual $\langle f \rangle$. □

3.3. Galois connections.

PROPOSITION 2264. The category of Galois connections between (small) lattices with least elements together with usual $\langle f \rangle$ defines a distributive system of presides.

PROOF. Propositions 2228 and 2229 for a system of presides.
It is distributive because lower adjoints preserve all joins. □

PROPOSITION 2265. The category of Galois connections between (small) bounded lattices together with usual $\langle f \rangle$ defines a bounded system of presides.

PROOF. Theorem 2232. □

PROPOSITION 2266. The category of Galois connections between (small) Heyting lattices together with usual $\langle f \rangle$ defines a system of sides with correct identities.

PROOF. Theorem 2233 ensures that they a system of sides with identities. The identities are correct due to faithfulness. □

3.4. Reloids.

PROPOSITION 2267. Reloids with the functor $f \mapsto \langle (\mathsf{FCD}) f \rangle$ and usual id_c form a system of sides with correct identities.

PROOF. It is really a functor because $\langle (\mathsf{FCD}) g \rangle \circ \langle (\mathsf{FCD}) f \rangle = \langle (\mathsf{FCD}) g \circ (\mathsf{FCD}) f \rangle = \langle (\mathsf{FCD})(g \circ f) \rangle$ for every composable reloids f and g.
$\langle a \rangle \bot = \langle (\mathsf{FCD}) a \rangle \bot = \bot$;

$$\langle a \sqcup b \rangle X = \langle (\mathsf{FCD})(a \sqcup b) \rangle X = \langle (\mathsf{FCD}) a \sqcup (\mathsf{FCD}) b) \rangle X = \langle (\mathsf{FCD}) a \rangle X \sqcup \langle (\mathsf{FCD}) b \rangle X = \langle a \rangle X \sqcup \langle b \rangle X;$$

thus it is a system of presides.

That this is a bounded system of presides follows from the formulas $(\mathsf{FCD})\bot^{\mathsf{RLD}(A,B)} = \bot$ and $(\mathsf{FCD})\top^{\mathsf{RLD}(A,B)} = \top$.

It is with identities, because proposition 1168. It is with correct identities by proposition 898. □

FiXme: Also for pointfree reloids.

FiXme: These examples works for (dagger) systems of sides with binary product.

4. Product

Definition 2268. *Binary product* of objects of presides with identities is defined by the formula $X \times Y = \mathrm{id}_Y \circ \top \circ \mathrm{id}_X$.

Definition 2269. System of presides with identities is *with correct binary product* when $f \sqcap (X \times Y) = \mathrm{id}_Y \circ f \circ \mathrm{id}_X$ for every preside f.

Proposition 2270. $\langle A \times B\rangle X = \begin{cases} \bot & \text{if } X \asymp A \\ B & \text{if } X \not\asymp A \end{cases}$

Proof.

$$\langle A \times B\rangle X = \langle \mathrm{id}_B \circ \top \circ \mathrm{id}_A\rangle X = \langle \mathrm{id}_B\rangle\langle\top\rangle\langle \mathrm{id}_A\rangle X = \\ B \sqcap \langle\top\rangle(X \sqcap A) = B \sqcap \begin{cases} \bot & \text{if } X \asymp A \\ \top & \text{if } X \not\asymp A \end{cases} = \begin{cases} \bot & \text{if } X \asymp A \\ B & \text{if } X \not\asymp A \end{cases}$$

□

Definition 2271. I will call a system of sides *with correct meet* when

$$(X_0 \times Y_0) \sqcap (X_1 \times Y_1) = (X_0 \sqcap X_1) \times (Y_0 \sqcap Y_1).$$

Proposition 2272. Faithful systems of presides with identities are with correct meet.

Proof. $(X_0 \times Y_0) \sqcap (X_1 \times Y_1) = \mathrm{id}_{Y_1} \circ (X_0 \times Y_0) \circ \mathrm{id}_{X_1}$. Thus

$$\langle (X_0 \times Y_0) \sqcap (X_1 \times Y_1)\rangle P = \langle \mathrm{id}_{Y_1}\rangle\langle X_0 \times Y_0\rangle\langle \mathrm{id}_{X_1}\rangle P = \\ \langle \mathrm{id}_{Y_1}\rangle \begin{cases} \bot & \text{if } X_0 \asymp \langle \mathrm{id}_{X_1}\rangle P \\ Y_0 & \text{if } X_0 \not\asymp \langle \mathrm{id}_{X_1}\rangle P \end{cases} = \begin{cases} \bot & \text{if } X_0 \sqcap X_1 \asymp P \\ Y_0 \sqcap Y_1 & \text{if } X_0 \sqcap X_1 \not\asymp P \end{cases} = \\ \langle (X_0 \sqcap X_1) \times (Y_0 \sqcap Y_1)\rangle P.$$

So $(X_0 \times Y_0) \sqcap (X_1 \times Y_1) = (X_0 \sqcap X_1) \times (Y_0 \sqcap Y_1)$ follows by full faithfulness. □

Proposition 2273. Systems of presides with correct identities are with correct meet.

Proof. $(X_0 \times Y_0) \sqcap (X_1 \times Y_1) = \mathrm{id}_{Y_1} \circ (X_0 \times Y_0) \circ \mathrm{id}_{X_1} = \mathrm{id}_{Y_1} \circ (\mathrm{id}_{Y_0} \circ \top \circ \mathrm{id}_{X_0}) \circ \mathrm{id}_{X_1} = \mathrm{id}_{Y_0 \sqcap Y_1} \circ \top \circ \mathrm{id}_{X_0 \sqcap X_1} = (X_0 \sqcap X_1) \times (Y_0 \sqcap Y_1)$. □

For some sides holds the formula $f \circ (X \times Y) = X \times \langle f\rangle Y$. I refrain to give a name for this property.

5. Negative results

The following negative result generalizes theorem 3.8 in [3].

THEOREM 2274. The element $1^{(\operatorname{Src}\Upsilon)(\mathfrak{A},\mathfrak{A})}$ is not complemented if $\mathfrak{A}$ is a non-atomic boolean lattice, for every monotone system of sides.

PROOF. Let $T = 1^{(\operatorname{Src}\Upsilon)(\mathfrak{A},\mathfrak{A})}$.

Let's suppose $T \sqcup V = \top$ for $V \in (\operatorname{Src}\Upsilon)(\mathfrak{A},\mathfrak{A})$ and prove $T \sqcap V \neq \bot$.

Then $\langle T \sqcup V\rangle a = \top$ for all $a \neq \bot$ and thus $\langle V\rangle a \sqcup a = \top$.

Consequently $\langle V\rangle a \sqsupseteq \neg a$ for all $a \neq \bot$.

If a isn't an atom, then there exists b with $0 \sqsubset b \sqsubset a$ and hence $\langle V\rangle a \sqsupseteq \langle V\rangle b \sqsupseteq \neg b \sqsupset \neg a$; thus $\langle V\rangle a \sqsupset \neg a$.

There is such $c \sqsubset \top$ that $a \sqsubseteq c$ for every atom a. (Really, suppose some element $p \neq \bot$ has no atoms. Thus all atoms are in $\neg p$.)

For $a \not\sqsubseteq c$ we have $\langle V\rangle a \sqcap a \sqsupset \bot$ for all $a \sqsubseteq \neg c$ thus $\langle T \sqcap V\rangle a \sqsupseteq \langle V\rangle a \sqcap a \sqsupset \bot$. Thus $\langle (T \sqcap V) \circ \mathrm{id}_{\neg c}\rangle a \sqsupset \bot$

So $T \sqcap V \sqsupseteq (T \sqcap V) \circ \mathrm{id}_{\neg c} \sqsupset \bot$. So V is not a complement of T. $\square$

COROLLARY 2275. $(\operatorname{Src}\Upsilon)(\mathfrak{A},\mathfrak{A})$ is not boolean if $\mathfrak{A}$ is a non-atomic boolean lattice.

6. Dagger systems of sides

PROPOSITION 2276.

1°. For a partially ordered dagger category, each of Hom-set of which has least element, we have $\bot^\dagger = \bot$.

2°. For a partially ordered dagger category, each of Hom-set of which has greatest element, we have $\top^\dagger = \top$.

PROOF. $\forall f \in \operatorname{Hom}(A,B) : \bot^\dagger \sqsubseteq f \Leftrightarrow \forall f \in \operatorname{Hom}(A,B) : \bot \sqsubseteq f^\dagger \Leftrightarrow \forall f \in \operatorname{Hom}(A,B) : \bot \sqsubseteq f \Leftrightarrow 1$. Thus $\bot^\dagger$ is the least.

The other items is dual. $\square$

DEFINITION 2277. *Dagger system of presides with identities* is system of presides with identities with category $\operatorname{Src}\Upsilon$ being a partially ordered dagger category and $(\mathrm{id}_X)^\dagger = \mathrm{id}_X$ for every X.

PROPOSITION 2278. For a system of sides we have $(X \times Y)^\dagger = Y \times X$.

PROOF. $(X \times Y)^\dagger = (\mathrm{id}_Y \circ \top \circ \mathrm{id}_X)^\dagger = \mathrm{id}_X^\dagger \circ \top^\dagger \circ \mathrm{id}_Y^\dagger = \mathrm{id}_X \circ \top \circ \mathrm{id}_Y = Y \times X$. $\square$

FiXme: Which properties of pointfree funcoids can be generalized for sides?

CHAPTER 9

Backward Funcoids

This is a preliminary partial draft.

Fix a family $\mathfrak{A}$ of posets.

Definition 2279. Let f be a staroid of filters $\mathfrak{F}(\mathfrak{A}_i)$ on boolean lattices $\mathfrak{A}_i$. *Backward funcoid* for the argument $k \in \operatorname{dom}\mathfrak{A}$ of f is the funcoid $\operatorname{Back}(f,k)$ defined by the formula (for every $X \in \mathfrak{A}_k$)

$$\langle \operatorname{Back}(f,k)\rangle X = \left\{ \frac{L \in \prod_{i\in\operatorname{dom}\mathfrak{A}} \mathfrak{F}(\mathfrak{A}_i)}{X \in \langle f\rangle_k L} \right\}.$$

Proposition 2280. Backward funcoid is properly defined.

Proof. $\langle \operatorname{Back}(f,k)\rangle^*(X \sqcup Y) = \left\{ \frac{L\in\prod\mathfrak{A}}{X\sqcup Y\in\langle f\rangle_k L} \right\} = \left\{ \frac{L\in\prod\mathfrak{A}}{X\in\langle f\rangle_k L \vee Y\in\langle f\rangle_k L} \right\} = \left\{ \frac{L\in\prod\mathfrak{A}}{X\in\langle f\rangle_k L} \right\} \cup \left\{ \frac{L\in\prod\mathfrak{A}}{Y\in\langle f\rangle_k L} \right\} = \langle \operatorname{Back}(f,k)\rangle^* X \cup \langle \operatorname{Back}(f,k)\rangle^* Y$. □

Obvious 2281. Backward funcoid is co-complete.

Proposition 2282. If f is a principal staroid then $\operatorname{Back}(f,k)$ is a complete funcoid.

Proof. ?? □

Proposition 2283. f can be restored from $\operatorname{Back}(f,k)$ (for every fixed k).

Proof. ?? □

Proposition 2284. $f \mapsto \operatorname{Back}(f,k)$ is an order isomorphism $\mathsf{Strd}^{\mathfrak{A}} \to \mathsf{FCD}\left(\mathfrak{A}_k, \mathsf{Strd}^{(\operatorname{dom}\mathfrak{A})\setminus\{k\}}\right)$.

Proof. ?? □

CHAPTER 10

Quasi-atoms

DEFINITION 2285. *Quasi-atoms* funcoid $\mathscr{A}$ is the funcoid $A \to \operatorname{atoms}^{\mathfrak{A}} A$ defined by the formula $\langle \mathscr{A} \rangle^* X = \operatorname{atoms}^{\mathfrak{A}} X$.

This really defines a funcoid because $\operatorname{atoms}^{\mathfrak{A}} \bot = \emptyset$ and $\operatorname{atoms}^{\mathfrak{A}}(X \cup Y) = \operatorname{atoms}^{\mathfrak{A}} X \cup \operatorname{atoms}^{\mathfrak{A}} Y$.

OBVIOUS 2286. $\mathscr{A}$ is a co-complete funcoid.

PROPOSITION 2287. $\left\langle \mathscr{A}^{-1} \right\rangle^* Y = \bigsqcup Y$.

PROOF. $Y \not\asymp \langle \mathscr{A} \rangle^* X \Leftrightarrow Y \not\asymp \operatorname{atoms}^{\mathfrak{A}} X \Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{A}} X, y \in Y : x \not\asymp y \Leftrightarrow \exists y \in Y : X \not\asymp y \Leftrightarrow$ (because X is a principal filter) $\Leftrightarrow X \not\asymp \bigsqcup Y$. □

Note $\langle \mathscr{A} \rangle^* \mathcal{X} = \bigsqcap^{\mathscr{F}}_{X \in \operatorname{up} \mathcal{X}} \operatorname{atoms}^{\mathfrak{A}} X$;
$\left\langle \mathscr{A}^{-1} \right\rangle^* \mathcal{Y} = \bigsqcap^{\mathscr{F}}_{Y \in \operatorname{up} \mathcal{Y}} \bigsqcup Y$ ($\mathcal{Y}$ is filter on the set of ultrafilters).
Can $\operatorname{atoms}^{\mathfrak{A}} \mathcal{X}$ be restored knowing $\langle \mathscr{A} \rangle \mathcal{X}$? Can $\bigsqcup \mathcal{Y}$ be restored knowing $\left\langle \mathscr{A}^{-1} \right\rangle \mathcal{X}$?

PROPOSITION 2288. (Provided that A is infinite) $\mathscr{A}$ is not complete.

PROOF. Take a nonprincipal ultrafilter x. Then $\left\langle \mathscr{A}^{-1} \right\rangle^* \{x\} = \bigsqcup \{x\} = x$ is a nonprincipal filter. □

CONJECTURE 2289. There is such filter $\mathcal{X}$ that $\langle \mathscr{A} \rangle^* \mathcal{X}$ is non-principal.

Does quasi-atoms funcoid define a more elegant replacement of $\operatorname{atoms}^{\mathfrak{A}}$? Does this concept have any use?

CHAPTER 11

Cauchy Filters on Reloids

In this chapter I consider *low filters* on reloids, generalizing Cauchy filters on uniform spaces. Using low filters, I define Cauchy-complete reloids, generalizing complete uniform spaces.

FiXme: I forgot to note that Cauchy spaces induce topological (or convergence) spaces.

1. Preface

Replace `\langle ...\rangle` with `\supfun{...}` in LaTeX.

This is a preliminary partial draft.

To understand this article you need first look into my book [2].

http://math.stackexchange.com/questions/401989/what-are-interesting-properties-of-totally-bounded-uniform-spaces

http://ncatlab.org/nlab/show/proximity+space#uniform_spaces for a proof sketch that proximities correspond to totally bounded uniformities.

2. Low spaces

FiXme: Analyze http://link.springer.com/article/10.1007/s10474-011-0136-9 ("A note on Cauchy spaces"), http://link.springer.com/article/10.1007/BF00873992 ("Filter spaces"). It also contains references to some useful results, including ("On continuity structures and spaces of mappings" freely available at https://eudml.org/doc/16128) that the category FIL of filter spaces is isomorphic to the category of filter merotopic spaces (copy its definition).

DEFINITION 2290. A *lower set*[1] of filters on U (a set) is a set $\mathscr{C}$ of filters on U, such that if $\mathcal{G} \sqsubseteq \mathcal{F}$ and $\mathcal{F} \in \mathscr{C}$ then $\mathcal{G} \in \mathscr{C}$.

REMARK 2291. Note that we are particularly interested in nonempty (= containing the improper filter) lower sets of filters. This does not match the traditional theory of Cauchy spaces (see below) which are traditionally defined as not containing empty set. Allowing them to contain empty set has some advantages:

- Meet of any lower filters is a lower filter.
- Some formulas become a little simpler.

DEFINITION 2292. I call *low space* a set together with a nonempty lower set of filters on this set. Elements of a (given) low space are called *Cauchy filters*.

DEFINITION 2293. $\mathrm{GR}(U, \mathscr{C}) = \mathscr{C}$; $\mathrm{Ob}(U, \mathscr{C}) = U$. $\mathrm{GR}(U, \mathscr{C})$ is read as *graph* of space $(U, \mathscr{C})$. I denote $\mathrm{Low}(U)$ the set of graphs of low spaces on the set U. Similarly I will denote its subsets $\mathbf{ASJ}(U)$, $\mathbf{CASJ}(U)$, $\mathbf{Cau}(U)$, $\mathbf{CCau}(U)$ (see below).

FiXme: Should use "space structure" instead of "graph of space", to match customary terminology.

[1] Remember that our orders on filters is the reverse to set theoretic inclusion. It could be called an *upper* set in other sources.

DEFINITION 2294. Introduce an order on graphs of low spaces and on low spaces: $\mathscr{C} \sqsubseteq \mathscr{D} \Leftrightarrow \mathscr{C} \subseteq \mathscr{D}$ and $(U, \mathscr{C}) \sqsubseteq (U, \mathscr{D}) \Leftrightarrow \mathscr{C} \sqsubseteq \mathscr{D}$.

OBVIOUS 2295. Every set of low spaces on some set is partially ordered.

3. Almost sub-join-semilattices

DEFINITION 2296. For a join-semilattice $\mathfrak{A}$, a *almost sub-join-semilattice* is such a set $S \in \mathscr{P}\mathfrak{A}$, that if $\mathcal{F}, \mathcal{G} \in S$ and $\mathcal{F} \not\asymp \mathcal{G}$ then $\mathcal{F} \sqcup \mathcal{G} \in S$.

DEFINITION 2297. For a complete lattice $\mathfrak{A}$, a *completely almost sub-join-semilattice* is such a set $S \in \mathscr{P}\mathfrak{A}$, that if $\bigsqcap T \neq \bot^{\mathscr{F}(X)}$ then $\bigsqcup T \in S$ for every $T \in \mathscr{P}S$.

OBVIOUS 2298. Every completely almost sub-join-semilattice is a almost sub-join-semilattice.

4. Cauchy spaces

DEFINITION 2299. A *reflexive* low space is a low space $(U, \mathscr{C})$ such that $\forall x \in U :\uparrow^U \{x\} \in \mathscr{C}$.

DEFINITION 2300. The *identity* low space $1^{\mathrm{Low}(U)}$ on a set U is the low space with graph $\left\{ \frac{\uparrow^U \{x\}}{x \in U} \right\}$.

OBVIOUS 2301. A low space f is reflexive iff $f \sqsupseteq 1^{\mathrm{Low}(\mathrm{Ob}\, f)}$.

DEFINITION 2302. *An almost sub-join space* is a low space whose graph is an almost sub-join-semilattice. I will denote such spaces as **ASJ**.

DEFINITION 2303. *A completely almost sub-join space* is a low space whose graph is a completely almost sub-join-semilattice. I will denote such spaces as **CASJ**.

DEFINITION 2304. A *precauchy space* (aka *filter space*) is a reflexive low space. I will denote such spaces as **preCau**.

DEFINITION 2305. A *Cauchy space* is a precauchy space which is also an almost sub-join space. I will denote such spaces as **Cau**.

DEFINITION 2306. A *completely Cauchy space* is a precauchy space which is also a completely almost sub-join space. I will denote such spaces as **CCau**.

OBVIOUS 2307. Every completely Cauchy space is a Cauchy space.

PROPOSITION 2308. $a \sqcup^{\left\{ \frac{\mathcal{X} \in \mathscr{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\}} b = a \sqcup b$ for $a, b \in \left\{ \frac{\mathcal{X} \in \mathscr{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\}$, provided that $\mathcal{F}$ is a proper fixed Cauchy filter on an almost sub-join space.

PROOF. $\mathcal{F}$ is proper. So we have $a \sqcap b \sqsupseteq \mathcal{F} \neq \bot^{\mathscr{F}(X)}$. Thus $a \sqcup b$ is a Cauchy filter and so $a \sqcup b \in \left\{ \frac{\mathcal{X} \in \mathscr{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\}$. □

PROPOSITION 2309. $\bigsqcup^{\left\{ \frac{\mathcal{X} \in \mathscr{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\}} S = \bigsqcup S$ for nonempty $S \in \mathscr{P}\left\{ \frac{\mathcal{X} \in \mathscr{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\}$, provided that $\mathcal{F}$ is a proper fixed Cauchy filter on a completely almost sub-join space.

PROOF. $\mathcal{F}$ is proper. So for every nonempty $S \in \mathscr{P}\left\{ \frac{\mathcal{X} \in \mathscr{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\}$ we have $\bigsqcap S \sqsupseteq \mathcal{F} \neq \bot^{\mathscr{F}(X)}$. Thus $\bigsqcup S$ is a Cauchy filter and so $\bigsqcup S \in \left\{ \frac{\mathcal{X} \in \mathscr{C}}{\mathcal{X} \sqsupseteq \mathcal{F}} \right\}$. □

COROLLARY 2310. Every proper Cauchy filter is contained in a unique maximal Cauchy filter (for completely almost sub-join spaces).

PROOF. Let $\mathcal{F}$ be a proper Cauchy filter. Then $\bigsqcup^{\left\{\frac{\mathcal{X}\in\mathscr{C}}{\mathcal{X}\sqsupseteq\mathcal{F}}\right\}}\left\{\frac{\mathcal{X}\in\mathscr{C}}{\mathcal{X}\sqsupseteq\mathcal{F}}\right\}$ (existing by the above proposition) is the maximal Cauchy filter containing $\mathcal{F}$.

Suppose another maximal Cauchy filter $\mathcal{T}$ contains $\mathcal{F}$. Then $\mathcal{T}\in\left\{\frac{\mathcal{X}\in\mathscr{C}}{\mathcal{X}\sqsupseteq\mathcal{F}}\right\}$ and thus $\mathcal{T}=\bigsqcup^{\left\{\frac{\mathcal{X}\in\mathscr{C}}{\mathcal{X}\sqsupseteq\mathcal{F}}\right\}}\left\{\frac{\mathcal{X}\in\mathscr{C}}{\mathcal{X}\sqsupseteq\mathcal{F}}\right\}$. □

5. Relationships with symmetric reloids

FiXme: Also consider relationships with funcoids.

DEFINITION 2311. Denote $(\mathsf{RLD})_{\mathrm{Low}}(U,\mathscr{C})=\bigsqcup\left\{\frac{\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}}{\mathcal{X}\in\mathscr{C}}\right\}$.

DEFINITION 2312. $(\mathrm{Low})\nu$ (*low space* for endoreloid ν) is a low space on U such that

$$\mathrm{GR}(\mathrm{Low})\nu=\left\{\frac{\mathcal{X}\in\mathscr{F}(U)}{\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}\sqsubseteq\nu}\right\}.$$

THEOREM 2313. If $(U,\mathscr{C})$ is a low space, then $(U,\mathscr{C})=(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}}(U,\mathscr{C})$.

PROOF. If $\mathcal{X}\in\mathscr{C}$ then $\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}\sqsubseteq(\mathsf{RLD})_{\mathrm{Low}}(U,\mathscr{C})$ and thus $\mathcal{X}\in\mathrm{GR}(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}}(U,\mathscr{C})$. Thus $(U,\mathscr{C})\sqsubseteq(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}}(U,\mathscr{C})$.

Let's prove $(U,\mathscr{C})\sqsupseteq(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}}(U,\mathscr{C})$.

Let $\mathcal{A}\in\mathrm{GR}(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}}(U,\mathscr{C})$. We need to prove $\mathcal{A}\in\mathscr{C}$.

Really $\mathcal{A}\times^{\mathsf{RLD}}\mathcal{A}\sqsubseteq(\mathsf{RLD})_{\mathrm{Low}}(U,\mathscr{C})$. It is enough to prove that $\exists\mathcal{X}\in\mathscr{C}:\mathcal{A}\sqsubseteq\mathcal{X}$.

Suppose $\nexists\mathcal{X}\in\mathscr{C}:\mathcal{A}\sqsubseteq\mathcal{X}$.

For every $\mathcal{X}\in\mathscr{C}$ obtain $X_{\mathcal{X}}\in\mathcal{X}$ such that $X_{\mathcal{X}}\notin\mathcal{A}$ (if forall $X\in\mathcal{X}$ we have $X_{\mathcal{X}}\in\mathcal{A}$, then $\mathcal{X}\sqsupseteq\mathcal{A}$ what is contrary to our supposition).

It is now enough to prove $\mathcal{A}\times^{\mathsf{RLD}}\mathcal{A}\not\sqsubseteq\bigsqcup\left\{\frac{\uparrow^U X_{\mathcal{X}}\times^{\mathsf{RLD}}\uparrow^U X_{\mathcal{X}}}{\mathcal{X}\in\mathscr{C}}\right\}$.

Really, $\bigsqcup\left\{\frac{\uparrow^U X_{\mathcal{X}}\times^{\mathsf{RLD}}\uparrow^U X_{\mathcal{X}}}{\mathcal{X}\in\mathscr{C}}\right\}=\uparrow^{\mathsf{RLD}(U,U)}\bigcup\left\{\frac{\uparrow^U X_{\mathcal{X}}\times^{\mathsf{RLD}}\uparrow^U X_{\mathcal{X}}}{\mathcal{X}\in\mathscr{C}}\right\}$. So our claim takes the form $\bigcup\left\{\frac{\uparrow^U X_{\mathcal{X}}\times^{\mathsf{RLD}}\uparrow^U X_{\mathcal{X}}}{\mathcal{X}\in\mathscr{C}}\right\}\notin\mathrm{GR}(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{A})$ that is $\forall A\in\mathcal{A}:\bigcup\left\{\frac{\uparrow^U X_{\mathcal{X}}\times^{\mathsf{RLD}}\uparrow^U X_{\mathcal{X}}}{\mathcal{X}\in\mathscr{C}}\right\}\not\supseteq A\times A$ what is true because $X_{\mathcal{X}}\not\supseteq A$ for every $A\in\mathcal{A}$. □

REMARK 2314. The last theorem does not hold with $\mathcal{X}\times^{\mathsf{FCD}}\mathcal{X}$ instead of $\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}$ (take $\mathscr{C}=\left\{\frac{\{x\}}{x\in U}\right\}$ for an infinite set U as a counter-example).

REMARK 2315. Not every symmetric reloid is in the form $(\mathsf{RLD})_{\mathrm{Low}}(U,\mathscr{C})$ for some Cauchy space $(U,\mathscr{C})$. The same Cauchy space can be induced by different uniform spaces. See http://math.stackexchange.com/questions/702182/different-uniform-spaces-having-the-same-set-of-cauchy-filters

PROPOSITION 2316.

1°. $(\mathrm{Low})f$ is reflexive iff endoreloid f is reflexive.
2°. $(\mathsf{RLD})_{\mathrm{Low}}f$ is reflexive iff low space f is reflexive.

PROOF.

1°. f is reflexive $\Leftrightarrow 1^{\mathsf{RLD}}\sqsubseteq f\Leftrightarrow\forall x\in\mathrm{Ob}\,f:\uparrow(\{x\}\times\{x\})\sqsubseteq f\Leftrightarrow\forall x\in\mathrm{Ob}\,f:\uparrow\{x\}\times^{\mathsf{RLD}}\uparrow\{x\}\sqsubseteq f\Leftrightarrow\forall x\in\mathrm{Ob}\,f:\uparrow\{x\}\in(\mathrm{Low})f\Leftrightarrow(\mathrm{Low})f$ is reflexive.

2°. Let f is reflexive. Then $\forall x\in\mathrm{Ob}\,f:\uparrow\{x\}\in f$; $\forall x\in\mathrm{Ob}\,f:\uparrow\{x\}\times^{\mathsf{RLD}}\uparrow\{x\}\sqsubseteq(\mathsf{RLD})_{\mathrm{Low}}f$; $\forall x\in\mathrm{Ob}\,f:\uparrow(\{x\}\times\{x\})\sqsubseteq(\mathsf{RLD})_{\mathrm{Low}}f$; $1^{\mathsf{RLD}}\sqsubseteq(\mathsf{RLD})_{\mathrm{Low}}f$.

Let now $(\mathsf{RLD})_{\mathrm{Low}}f$ be reflexive. Then $f=(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}}f$ is reflexive. □

DEFINITION 2317. A *transitive* low space is such low space f that $(\mathsf{RLD})_{\mathrm{Low}} f \circ (\mathsf{RLD})_{\mathrm{Low}} f = (\mathsf{RLD})_{\mathrm{Low}} f$.

REMARK 2318. The composition $(\mathsf{RLD})_{\mathrm{Low}} f \circ (\mathsf{RLD})_{\mathrm{Low}} f$ may be inequal to $(\mathsf{RLD})_{\mathrm{Low}} \mu$ for all low spaces μ (exercise!). Thus usefulness of the concept of transitive low spaces is questionable.

REMARK 2319. Every low space is "symmetric" in the sense that it corresponds to a symmetric reloid.

THEOREM 2320. (Low) is the upper adjoint of $(\mathsf{RLD})_{\mathrm{Low}}$.

PROOF. We will prove $(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}} f \sqsupseteq f$ and $(\mathsf{RLD})_{\mathrm{Low}}(\mathrm{Low}) g \sqsubseteq g$ (that (Low) and $(\mathsf{RLD})_{\mathrm{Low}}$ are monotone is obvious).

Really:

$$\mathrm{GR}(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}} f = \mathrm{GR}(\mathrm{Low}) \bigsqcup \left\{ \frac{\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X}}{\mathcal{X} \in \mathrm{GR}\, f} \right\} = \left\{ \frac{\mathcal{Y} \in \mathscr{F} \mathrm{Ob}(f)}{\mathcal{Y} \times^{\mathsf{RLD}} \mathcal{Y} \sqsubseteq \bigsqcup \left\{ \frac{\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X}}{\mathcal{X} \in \mathrm{GR}\, f} \right\}} \right\} \supseteq \left\{ \frac{\mathcal{Y} \in \mathscr{F} \mathrm{Ob}(f)}{\mathcal{Y} \in \mathrm{GR}\, f} \right\} = \mathrm{GR}\, f;$$

$$(\mathsf{RLD})_{\mathrm{Low}}(\mathrm{Low}) g = \bigsqcup \left\{ \frac{\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X}}{\mathcal{X} \in \mathrm{GR}(\mathrm{Low}) g} \right\} = \bigsqcup \left\{ \frac{\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X}}{\mathcal{X} \in \mathscr{F}(\mathrm{Ob}\, g), \mathcal{X} \times^{\mathsf{RLD}} \mathcal{X} \sqsubseteq g} \right\} \sqsubseteq g.$$

□

COROLLARY 2321.

1°. $(\mathsf{RLD})_{\mathrm{Low}} \bigsqcup S = \bigsqcup \langle (\mathsf{RLD})_{\mathrm{Low}} \rangle^* S$;
2°. $(\mathrm{Low}) \bigsqcap S = \bigsqcap \langle (\mathrm{Low}) \rangle^* S$.

Below it's proved that (Low) and $(\mathsf{RLD})_{\mathrm{Low}}$ can be restricted to completely almost sub-join spaces and symmetrically transitive reloids. Thus they preserve joins of (completely) almost sub-join spaces and meets of symmetrically transitive reloids. FiXme: Check. FiXme: Move it to be below the definition.

6. Lattices of low spaces

PROPOSITION 2322. $\mu \sqsubseteq \nu \Leftrightarrow \forall \mathcal{X} \in \mathrm{GR}\, \mu \exists \mathcal{Y} \in \mathrm{GR}\, \nu : \mathcal{X} \sqsubseteq \mathcal{Y}$ for low filter spaces (on the same set U).

PROOF.

$\Rightarrow$. $\mu \sqsubseteq \nu \Leftrightarrow \mathrm{GR}\, \mu \subseteq \mathrm{GR}\, \nu \Rightarrow \forall \mathcal{X} \in \mathrm{GR}\, \mu \exists \mathcal{Y} \in \mathrm{GR}\, \nu : \mathcal{X} = \mathcal{Y} \Rightarrow \forall \mathcal{X} \in \mathrm{GR}\, \mu \exists \mathcal{Y} \in \mathrm{GR}\, \nu : \mathcal{X} \sqsubseteq \mathcal{Y}$.

$\Leftarrow$. Let $\forall \mathcal{X} \in \mathrm{GR}\, \mu \exists \mathcal{Y} \in \mathrm{GR}\, \nu : \mathcal{X} \sqsubseteq \mathcal{Y}$. Take $\mathcal{X} \in \mathrm{GR}\, \mu$. Then $\exists \mathcal{Y} \in \mathrm{GR}\, \nu : \mathcal{X} \sqsubseteq \mathcal{Y}$. Thus $\mathcal{X} \in \mathrm{GR}\, \nu$. So $\mathrm{GR}\, \mu \subseteq \mathrm{GR}\, \nu$ that is $\mu \sqsubseteq \nu$.

□

OBVIOUS 2323.

1°. $(\mathsf{RLD})_{\mathrm{Low}}$ is an order embedding.
2°. (Low) is an order homomorphism.

I will denote $\bigsqcup$, $\bigsqcap$, $\sqcup$, $\sqcap$ the lattice operations on low spaces or graphs of low spaces.

PROPOSITION 2324. $\bigsqcup S = \bigcup S$ for every set S of graphs of low spaces on some set.

PROOF. It's enough to prove that there is a low space μ such that $\mathrm{GR}\, \mu = \bigcup S$. In other words, it's enough to prove that $\bigcup S$ is a nonempty lower set, but that's obvious. FiXme: A little more detailed proof. □

Proposition 2325. $\prod S = \left\{ \frac{\prod \operatorname{im} P}{P \in \prod_{X\in S} X} \right\}$ for every set S of graphs of low spaces on some set.

Proof. First prove that there is such low space μ that $\mu = \left\{ \frac{\prod \operatorname{im} P}{P \in \prod_{X\in S} X} \right\}$. In other words, we need to prove that $\left\{ \frac{\prod \operatorname{im} P}{P \in \prod_{X\in S} X} \right\}$ is a nonempty lower set. That it is nonempty is obvious. Let filter $\mathcal{G} \sqsubseteq \mathcal{F}$ and $\mathcal{F} \in \left\{ \frac{\prod \operatorname{im} P}{P \in \prod_{X\in S} X} \right\}$. Then $\mathcal{F} = \prod \operatorname{im} P$ for a $P \in \prod_{X\in S} X$ that is $P(X) \in X$ for every $X \in S$. Take $P' = (\mathcal{G}\sqcap) \circ P$. Then $P' \in \prod_{X\in S} X$ because $P'(X) \in X$ for every $X \in S$ and thus obviously $\mathcal{G} = \prod \operatorname{im} P'$ and thus $\mathcal{G} \in \left\{ \frac{\prod \operatorname{im} P}{P \in \prod_{X\in S} X} \right\}$. So such μ exists.

It remains to prove that μ is the greatest lower bound of S.

μ is a lower bound of S. Really, let $X \in S$ and $Y \in X$. Then exists $P \in \prod_{X\in S} X$ such that $P(X) = Y$ (taken into account that every X is nonempty) and thus $\operatorname{im} P \ni Y$ and so $\prod \operatorname{im} P \sqsubseteq Y$, that is (proposition 2322) $\mu \sqsubseteq X$.

Let ν be a lower bound of S. It remains to prove that $\mu \sqsupseteq \nu$, that is $\forall Q \in \nu :$ $Q = \prod \operatorname{im} P$ for some $P \in \prod_{X\in S} X$. Take $P = (\lambda X \in S : Q)$. This $P \in \prod_{X\in S} X$ because $Q \in X$ for every $X \in S$. □

Corollary 2326. $f \sqcap g = \left\{ \frac{F \sqcap G}{F \in f, G \in g} \right\}$ for every graphs f and g of low spaces (on some set).

6.1. Its subsets.

Proposition 2327. The set of sub-join low spaces (on some fixed set) is meet-closed in the lattice of low spaces on a set.

Proof. Let f, g be graphs of almost sub-join spaces (on some fixed set), $f \sqcap g = \left\{ \frac{F \sqcap G}{F \in f, G \in g} \right\}$.

If $\mathcal{A}, \mathcal{B} \in f \sqcap g$ and $\mathcal{A} \not\asymp \mathcal{B}$, then $\mathcal{A}, \mathcal{B} \in f$ and $\mathcal{A}, \mathcal{B} \in g$. Thus $\mathcal{A} \sqcup \mathcal{B} \in f$ and $\mathcal{A} \sqcup \mathcal{B} \in g$ and so $\mathcal{A} \sqcup \mathcal{B} \in f \sqcap g$. □

Corollary 2328. The set of Cauchy spaces (on some fixed set), is meet-closed in the lattice of low spaces on a set.

Proposition 2329. The set of completely almost sub-join spaces is meet-closed in the lattice of low spaces on a set.

Proof. Let S be a set of graphs of almost completely sub-join low spaces (on some fixed set). $\prod S = \left\{ \frac{\prod \operatorname{im} P}{P \in \prod_{X\in S} X} \right\}$.

If $\mathcal{A}, \mathcal{B} \in \prod S$ and $\mathcal{A} \not\asymp \mathcal{B}$, then $\mathcal{A}, \mathcal{B} \in X$ for every $X \in S$. Thus $\mathcal{A} \sqcup \mathcal{B} \in X$ and so $\mathcal{A} \sqcup \mathcal{B} \in \prod S$. □

Corollary 2330. The set of completely Cauchy spaces is meet-closed in the lattice of low spaces on a set.

From the above it follows:

Obvious 2331. The following sets are complete lattices in our order:

1°. almost sub-join spaces, whose graphs are almost sub-join-semilattices;
2°. completely almost sub-join spaces;
3°. reflexive low spaces;
4°. precauchy spaces;

5°. Cauchy spaces;
6°. completely Cauchy spaces.

Denote $Z(f) = \left\{ \frac{F \sqcup G}{F \in f, G \in f, F \not\asymp G} \right\} \cup \{\bot\}$ for every set f of filters (on some fixed set).

PROPOSITION 2332. $Z(f) \sqsupseteq f$ for every set f of filters.

PROOF. Consider for $F \in f$ both cases $F = \bot$ and $F \neq \bot$. □

LEMMA 2333. For graphs of low spaces f, g (on the same set)

$$Q = \bigcup S \cup Z\Big(\bigcup S\Big) \cup Z\Big(Z\Big(\bigcup S\Big)\Big) \cup \dots$$

is a graph of some almost sub-join space.

PROOF. That it is nonempty and a lower set of filters is obvious. It remains to prove that it is an almost sub-join-semilattice.

Let $\mathcal{A}, \mathcal{B} \in Q$ and $\mathcal{A} \not\asymp \mathcal{B}$. Then

$$\mathcal{A}, \mathcal{B} \in \underbrace{Z \dots Z}_{n \text{ times}}\Big(\bigcup S\Big)$$

for a natural n. Thus

$$\mathcal{A} \sqcup \mathcal{B} \in \underbrace{Z \dots Z}_{n+1 \text{ times}}\Big(\bigcup S\Big)$$

and so $\mathcal{A} \sqcup \mathcal{B} \in Q$. □

PROPOSITION 2334. Join on the lattice of graphs of almost sub-join spaces is described by the formula

$$\bigsqcup^{\mathbf{ASJ}} S = \bigcup S \cup Z\Big(\bigcup S\Big) \cup Z\Big(Z\Big(\bigcup S\Big)\Big) \cup \dots$$

PROOF. The right part of the above formula μ is a graph of an almost sub-join space (lemma).

That μ is an upper bound of S is obvious.

It remains to prove that μ is the least upper bound.

Suppose ν is an upper bound of S. Then $\nu \supseteq \bigcup S$. Thus, because ν is an almost sub-join-semilattice, $Z(\nu) \subseteq \nu$, likewise $Z(Z(\nu)) \subseteq \nu$, etc. Consequently $Z(\bigcup S) \subseteq \nu$, $Z(Z(\bigcup S)) \subseteq \nu$, etc. So we have $\mu \sqsubseteq \nu$. □

PROPOSITION 2335. FiXme: Should be merged with the previous proposition.

$$\bigsqcup^{\mathbf{ASJ}} S = \left\{ \frac{F_0 \sqcup \dots \sqcup F_{n-1}}{F_0, \dots, F_{n-1} \in \bigcup S, F_0 \not\asymp F_1 \wedge F_1 \not\asymp F_2 \wedge \dots \wedge F_{n-2} \not\asymp F_{n-1} \text{ for } n \in \mathbb{N}} \right\}.$$

REMARK 2336. We take $F_0 \sqcup \dots \sqcup F_{n-1} = \bot$ for $n = 0$.

PROOF. Denote the right part of the above formula as R.

Suppose $F \in R$. Let's prove by induction that $F \in Q$. If $F = \bot$ that's obvious. Suppose we know that $F_0 \sqcup \dots \sqcup F_{n-1} \in Q$ that is for a natural m

$$F_0 \sqcup \dots \sqcup F_{n-1} \in \underbrace{Z \dots Z}_{m \text{ times}}\Big(\bigcup S\Big)$$

for $F_0, \dots, F_{n-1} \in \bigcup S$, $F_0 \not\asymp F_1 \wedge F_1 \not\asymp F_2 \wedge \dots \wedge F_{n-2} \not\asymp F_{n-1}$ and also $F_{n-1} \not\asymp F_n$. Then $F_0 \sqcup \dots \sqcup F_{n-1} \not\asymp F_n$ and thus $F_0 \sqcup \dots \sqcup F_{n-1} \sqcup F_n \in \underbrace{Z \dots Z}_{m+1 \text{ times}}(\bigcup S)$ that is $F_0 \sqcup \dots \sqcup F_{n-1} \sqcup F_n \in Q$. So $F \in Q$ for every $F \in R$.

Now suppose $F \in Q$ that is for a natural m
$$F \in \underbrace{Z \ldots Z}_{m \text{ times}}\left(\bigcup S\right).$$
Let's prove by induction that $F = F_0 \sqcup \cdots \sqcup F_{n-1}$ for some $F_0, \ldots, F_{n-1} \in \bigcup S$ such that $F_0 \not\asymp F_1 \wedge F_1 \not\asymp F_2 \wedge \cdots \wedge F_{n-2} \not\asymp F_{n-1}$. If $m = 0$ then $F \in \bigcup S$ and our promise is obvious. Let our statement holds for a natural m. Prove that it holds for
$$F' \in \underbrace{Z \ldots Z}_{m+1 \text{ times}}\left(\bigcup S\right).$$
We have $F' = Z(F)$ for some $F = F_0 \sqcup \cdots \sqcup F_{n-1}$ where $F_0 \not\asymp F_1 \wedge F_1 \not\asymp F_2 \wedge \cdots \wedge F_{n-2} \not\asymp F_{n-1}$. The case $F' = \bot$ is easy. So we can assume $F' = A \sqcup B$ where $A, B \in F$ and $A \not\asymp B$. By the statement of induction $A = A_0 \sqcup \cdots \sqcup A_{p-1}$, $B = B_0 \sqcup \cdots \sqcup B_{q-1}$ for natural p and q, where $A_0 \not\asymp A_1 \wedge A_1 \not\asymp A_2 \wedge \cdots \wedge A_{p-2} \not\asymp A_{p-1}$, $B_0 \not\asymp B_1 \wedge B_1 \not\asymp B_2 \wedge \cdots \wedge B_{n-2} \not\asymp B_{n-1}$. Take j such that $A \not\asymp B_j$ and then take i such that $A_i \not\asymp B_j$. Then (using symmetry of the relation $\not\asymp$) we have $(A_0 \not\asymp A_1 \wedge A_1 \not\asymp A_2 \wedge \cdots \wedge A_{p-2} \not\asymp A_{p-1}) \wedge (A_{p-1} \not\asymp A_{p-2} \not\asymp \ldots A_{i+1} \not\asymp A_i) \wedge A_i \not\asymp B_j \wedge (B_j \not\asymp B_{j-1} \wedge \cdots \wedge B_1 \not\asymp B_0) \wedge (B_0 \not\asymp B_1 \wedge B_1 \not\asymp B_2 \wedge \cdots \wedge B_{q-2} \not\asymp B_{q-1})$. So $F' = A \sqcup B$ is representable as the join of a finite sequence of filters with each adjacent pair of filters in this sequence being intersecting. That is $F' \in Q$. □

PROPOSITION 2337. The lattice of Cauchy spaces (on some set) is a complete sublattice of the lattice of almost sub-join spaces.

PROOF. It's obvious, taking into account obvious 2301. □

Denote $Z_\infty(f) = \left\{ \frac{\bigsqcup T}{T \in \mathscr{P} f \wedge \bigsqcap T \neq \bot} \right\} \cup \{\bot\}$.

PROPOSITION 2338. $Z_\infty(f) \sqsupseteq f$.

PROOF. Consider for $F \in f$ both cases $F = \bot$ and $F \neq \bot$. □

LEMMA 2339. If S is a set of graphs of low spaces, then
$$Q = \bigcup S \cup Z_\infty\left(\bigcup S\right) \cup Z_\infty\left(Z_\infty\left(\bigcup S\right)\right) \cup \ldots$$
is a graph of a completely Cauchy space.

PROOF. That it is nonempty and a lower set of filters is obvious. It remains to prove that it is a completely almost sub-join-semilattice.

Let $T \in \mathscr{P} Q$ and $\bigsqcap T \neq \bot$. Then
$$T \in \mathscr{P} \underbrace{Z_\infty \ldots Z_\infty}_{n \text{ times}}\left(\bigcup S\right)$$
for a natural n. Thus
$$T \in \mathscr{P} \underbrace{Z_\infty \ldots Z_\infty}_{n+1 \text{ times}}\left(\bigcup S\right)$$
and so $\bigsqcup T \in Q$. □

PROPOSITION 2340. The lattice of completely Cauchy spaces (on some set) is a complete sublattice of the lattice of completely almost sub-join spaces.

PROOF. It's obvious, taking into account obvious 2301. □

PROPOSITION 2341. Join of a set S on the lattice of graphs of completely almost sub-join-semilattice is described by the formula:
$$\bigsqcup^{\mathbf{CASJ}} S = \bigcup S \cup Z_\infty\left(\bigcup S\right) \cup Z_\infty\left(Z_\infty\left(\bigcup S\right)\right) \cup \ldots$$

PROOF. The right part of the above formula μ is a graph of an almost sub-join space (lemma).

That μ is an upper bound of S is obvious.

It remains to prove that μ is the least upper bound.

Suppose ν is an upper bound of S. Then $\nu \supseteq \bigcup S$. Thus, because ν is an almost sub-join-semilattice, $Z_\infty(\nu) \subseteq \nu$, likewise $Z_\infty(Z_\infty(\nu)) \subseteq \nu$, etc. Consequently $Z_\infty(\bigcup S) \subseteq \nu$, $Z_\infty(Z_\infty(\bigcup S)) \subseteq \nu$, etc. So we have $\mu \sqsubseteq \nu$. □

CONJECTURE 2342.

1°. $$\bigsqcup^{\mathbf{CASJ}} S = \left\{ \frac{\bigsqcup T_0 \sqcup \dots \sqcup \bigsqcup T_{n-1}}{\begin{array}{l} n \in \mathbb{N}, T_0, \dots, T_{n-1} \in \bigcup S, \\ \bigsqcap T_0 \neq \bot \wedge \dots \wedge \bigsqcap T_{n-1} \neq \bot, \\ \bigsqcup T_0 \not\asymp \bigsqcup T_1 \wedge \dots \wedge \bigsqcup T_{n-2} \not\asymp \bigsqcup T_{n-1}. \end{array}} \right\};$$

2°. $$\bigsqcup^{\mathbf{CASJ}} S = \left\{ \frac{\bigsqcup T_0 \sqcup \bigsqcup T_1 \sqcup \dots}{\begin{array}{l} T_0, T_1, \dots \in \bigcup S, \\ \bigsqcap T_0 \neq \bot \wedge \bigsqcap T_2 \neq \bot \wedge \dots, \bigsqcup T_0 \not\asymp \bigsqcup T_1 \wedge \bigsqcup T_1 \not\asymp \bigsqcup T_2 \wedge \dots \end{array}} \right\}.$$

7. Up-complete low spaces

DEFINITION 2343. *Ideal base* is a nonempty subset S of a poset such that $\forall a, b \in S \exists c \in S : (a, b \sqsubseteq c)$.

OBVIOUS 2344. Ideal base is dual of filter base.

THEOREM 2345. Product of nonempty posets is a ideal base iff every factor is an ideal base.

PROOF. FiXme: more detailed proof

In one direction it is easy: Suppose one multiplier is not a dcpo. Take a chain with fixed elements (thanks our posets are nonempty) from other multipliers and for this multiplier take the values which form a chain without the join. This proves that the product is not a dcpo.

Let now every factor is dcpo. S is a filter base in $\prod \mathfrak{A}$ iff each component is a filter base. Each component has a join. Thus by proposition 638 S has a componentwise join. □

DEFINITION 2346. I call a low space *up-complete* when each ideal base (or equivalently every nonempty chain, see theorem 586) in this space has join in this space.

REMARK 2347. Elements of this ideal base are filters. (Thus is could be called a generalized ideal base.)

EXAMPLE 2348.

1°. $\left\{ \frac{\mathcal{X} \in \mathfrak{F}]0; +\infty[}{\exists \varepsilon > 0 : \mathcal{X} \sqsubseteq \uparrow]\varepsilon; +\infty[} \right\} \cup \uparrow \{0\}$ is a graph of Cauchy space on $\mathbb{R}_+$, but not up-complete.

2°. $\mathfrak{F}[0; +\infty[$ is a strictly greater graph of Cauchy space on $\mathbb{R}_+$ and is up-complete.

Lemma 2349. Let f be a reloid. Each ideal base $T \subseteq \left\{ \frac{(\mathcal{A},\mathcal{B})}{\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}\sqsubseteq f} \right\}$ has a join in this set.

Proof. Let T be an ideal base and $\forall(\mathcal{A},\mathcal{B}) \in T : \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \sqsubseteq f$.
$\forall(\mathcal{A},\mathcal{B}) \in T \forall \mathcal{X} \in \mathscr{F} \operatorname{Src} f : (\mathcal{X} \not\asymp \mathcal{A} \Rightarrow \mathcal{B} \sqsubseteq \langle f \rangle \mathcal{X})$;
taking join we have:
$\forall \mathcal{A} \in \operatorname{Pr}_0 T \forall \mathcal{X} \in \mathscr{F} \operatorname{Src} f : \left(\mathcal{X} \not\asymp \mathcal{A} \Rightarrow \bigsqcup_{\mathcal{B}\in\operatorname{Pr}_1 T} \mathcal{B} \sqsubseteq \langle f \rangle \mathcal{X}\right)$;
$\forall \mathcal{A} \in \operatorname{Pr}_0 T : \mathcal{A} \times^{\mathsf{FCD}} \bigsqcup_{\mathcal{B}\in\operatorname{Pr}_1 T} \mathcal{B} \sqsubseteq f$.
Now repeat a similar operation second time:
$\forall \mathcal{A} \in \operatorname{Pr}_0 T : \bigsqcup_{\mathcal{B}\in\operatorname{Pr}_1 T} \mathcal{B} \times^{\mathsf{FCD}} \mathcal{A} \sqsubseteq f^{-1}$;
$\forall \mathcal{A} \in \operatorname{Pr}_0 T \forall \mathcal{Y} \in \mathscr{F} \operatorname{Dst} f : \left(\mathcal{Y} \not\asymp \bigsqcup_{\mathcal{B}\in\operatorname{Pr}_1 T} \mathcal{B} \Rightarrow \mathcal{A} \sqsubseteq \langle f^{-1} \rangle \mathcal{Y}\right)$;
$\forall \mathcal{Y} \in \mathscr{F} \operatorname{Dst} f : \left(\mathcal{Y} \not\asymp \bigsqcup_{\mathcal{B}\in\operatorname{Pr}_1 T} \mathcal{B} \Rightarrow \bigsqcup_{\mathcal{A}\in\operatorname{Pr}_0 T} \mathcal{A} \sqsubseteq \langle f^{-1} \rangle \mathcal{Y}\right)$;
$\bigsqcup_{\mathcal{B}\in\operatorname{Pr}_1 T} \mathcal{B} \times^{\mathsf{FCD}} \bigsqcup_{\mathcal{A}\in\operatorname{Pr}_0 T} \mathcal{A} \sqsubseteq f^{-1}$;
$\bigsqcup_{\mathcal{A}\in\operatorname{Pr}_0 T} \mathcal{A} \times^{\mathsf{FCD}} \bigsqcup_{\mathcal{B}\in\operatorname{Pr}_1 T} \mathcal{B} \sqsubseteq f$. But $\bigsqcup_{\mathcal{A}\in\operatorname{Pr}_0 T} \mathcal{A} \times^{\mathsf{FCD}} \bigsqcup_{\mathcal{B}\in\operatorname{Pr}_1 T} \mathcal{B}$ is the join in consideration, because ideal base is ideal base in each argument. □

Proposition 2350. A Cauchy space generated by an endoreloid is always up-complete.

Proof. Let f be an endoreloid. $\operatorname{GR}(\operatorname{Low}) f = \left\{ \frac{\mathcal{X}\in\operatorname{Ob} f}{\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}\sqsubseteq f} \right\}$.
Let $T \subseteq \left\{ \frac{\mathcal{X}\in\operatorname{Ob} f}{\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}\sqsubseteq f} \right\}$ be an ideal base.

Then $N = \left\{ \frac{(\mathcal{F},\mathcal{F})}{\mathcal{F}\in T} \right\}$ is also an ideal base. Obviously $N \subseteq \left\{ \frac{(\mathcal{A},\mathcal{B})}{\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}\sqsubseteq f} \right\}$. Thus by the lemma it has a join in $\left\{ \frac{(\mathcal{A},\mathcal{B})}{\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}\sqsubseteq f} \right\}$. It's easy to see that this join is in $\left\{ \frac{(\mathcal{A},\mathcal{A})}{\mathcal{A}\in\operatorname{Ob} f, \mathcal{A}\times^{\mathsf{RLD}}\mathcal{A}\sqsubseteq f} \right\}$. Consequently T has a join in $\left\{ \frac{\mathcal{X}\in\operatorname{Ob} f}{\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}\sqsubseteq f} \right\}$. □

It is long time known that (using our terminology) low space induced by a uniform space is a Cauchy space, but that it is complete and up-complete is probably first discovered by Victor Porton.

8. More on Cauchy filters

Obvious 2351. Low filter on an endoreloid ν is a filter $\mathcal{F}$ such that

$$\forall U \in \operatorname{GR} f \exists A \in \mathcal{F} : A \times A \subseteq U.$$

Remark 2352. The above formula is the standard definition of Cauchy filters on uniform spaces.

Proposition 2353. If $\nu \sqsupseteq \nu \circ \nu^{-1}$ then every neighborhood filter is a Cauchy filter, that it

$$\nu \sqsupseteq \langle (\mathsf{FCD})\nu \rangle^* \{x\} \times^{\mathsf{RLD}} \langle (\mathsf{FCD})\nu \rangle^* \{x\}$$

for every point x.

Proof. $\langle (\mathsf{FCD})\nu \rangle^* \{x\} \times^{\mathsf{RLD}} \langle (\mathsf{FCD})\nu \rangle^* \{x\} = \langle (\mathsf{FCD})\nu \rangle \uparrow^{\operatorname{Ob}\nu} \{x\} \times^{\mathsf{RLD}} \langle (\mathsf{FCD})\nu \rangle \uparrow^{\operatorname{Ob}\nu} \{x\} = \nu \circ (\uparrow^{\operatorname{Ob}\nu} \{x\} \times^{\mathsf{RLD}} \uparrow^{\operatorname{Ob}\nu} \{x\}) \circ \nu^{-1} = \nu \circ (\uparrow^{\mathsf{RLD}(\operatorname{Ob}\nu,\operatorname{Ob}\nu)} \{(x,x)\}) \circ \nu^{-1} \sqsubseteq \nu \circ \operatorname{id}^{\mathsf{RLD}(\operatorname{Ob}\nu,\operatorname{Ob}\nu)} \circ \nu^{-1} = \nu \circ \nu^{-1} \sqsubseteq \nu$. □

Proposition 2354. If $\nu \sqsupseteq \nu \circ \nu^{-1}$ a filter converges (in ν) to a point, it is a low filter, provided that every neighborhood filter is a low filter.

Proof. Let $\mathcal{F} \sqsubseteq \langle (\mathsf{FCD})\nu \rangle^* \{x\}$. Then $\mathcal{F} \times^{\mathsf{RLD}} \mathcal{F} \sqsubseteq \langle (\mathsf{FCD})\nu \rangle^* \{x\} \times^{\mathsf{RLD}} \langle (\mathsf{FCD})\nu \rangle^* \{x\} \sqsubseteq \nu$. □

Corollary 2355. If a filter converges to a point, it is a low filter, provided that $\nu \sqsupseteq \nu \circ \nu^{-1}$.

9. Maximal Cauchy filters

LEMMA 2356. Let S be a set of sets with $\bigsqcap\langle\uparrow^{\mathfrak{F}}\rangle^* S \neq 0^{\mathfrak{F}}$ (in other words, S has finite intersection property). Let $T = \left\{ \frac{X \times X}{X \in S} \right\}$. Then

$$\bigcup T \circ \bigcup T = \bigcup S \times \bigcup S.$$

PROOF. Let $x \in \bigcup S$. Then $x \in X$ for some $X \in S$. $\langle \bigcup T\rangle\{x\} \sqsupseteq \uparrow X \supseteq \bigcap S \neq \emptyset$. Thus $\langle \bigcup T \circ \bigcup T\rangle\{x\} = \langle \bigcup T\rangle\langle \bigcup T\rangle\{x\} \in \left\langle \uparrow^{\mathsf{FCD}} \bigcup T\right\rangle \bigsqcap\langle\uparrow^{\mathfrak{F}}\rangle S \sqsupseteq \bigsqcup\left\{ \frac{\langle\uparrow^{\mathsf{FCD}}(X \times X)\rangle \bigsqcap\langle\uparrow^{\mathfrak{F}}\rangle S}{X \in S} \right\} = \bigsqcup\left\{ \frac{\uparrow^{\mathfrak{F}} X}{X \in S} \right\} = \bigsqcup\langle\uparrow^{\mathfrak{F}}\rangle S$ that is $\langle \bigcup T \circ \bigcup T\rangle\{x\} \supseteq \bigcup S$. □

COROLLARY 2357. Let S be a set of filters (on some fixed set) with nonempty meet. Let

$$T = \left\{ \frac{\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X}}{\mathcal{X} \in S} \right\}$$

Then

$$\bigsqcup T \circ \bigsqcup T = \bigsqcup S \times^{\mathsf{RLD}} \bigsqcup S.$$

PROOF. $\bigsqcup T \circ \bigsqcup T = \bigsqcap\left\{ \frac{\uparrow^{\mathfrak{F}}(X \circ X)}{X \in \bigsqcup T} \right\}$.

If $X \in \bigsqcup T$ then $X = \bigcup_{Q \in T}(P_Q \times P_Q)$ where $P_Q \in Q$. Therefore by the lemma we have

$$\bigcup\left\{ \frac{P_Q \times P_Q}{Q \in T} \right\} \circ \bigcup\left\{ \frac{P_Q \times P_Q}{Q \in T} \right\} = \bigcup_{Q \in T} P_Q \times \bigcup_{Q \in T} P_Q.$$

Thus $X \circ X = \bigcup_{Q \in T} P_Q \times \bigcup_{Q \in T} P_Q$.

Consequently $\bigsqcup T \circ \bigsqcup T = \bigsqcap\left\{ \frac{\uparrow^{\mathfrak{F}}\left(\bigcup_{Q \in T} P_Q \times \bigcup_{Q \in T} P_Q\right)}{X \in \bigsqcup T} \right\} \sqsupseteq \bigsqcup S \times^{\mathsf{RLD}} \bigsqcup S$.

$\bigsqcup T \circ \bigsqcup T \sqsubseteq \bigsqcup S \times^{\mathsf{RLD}} \bigsqcup S$ is obvious. □

DEFINITION 2358. I call an endoreloid ν *symmetrically transitive* iff for every symmetric endofuncoid $f \in \mathsf{FCD}(\operatorname{Ob}\nu, \operatorname{Ob}\nu)$ we have $f \sqsubseteq \nu \Rightarrow f \circ f \sqsubseteq \nu$.

OBVIOUS 2359. It is symmetrically transitive if at least one of the following holds:

1°. $\nu \circ \nu \sqsubseteq \nu$;
2°. $\nu \circ \nu^{-1} \sqsubseteq \nu$;
3°. $\nu^{-1} \circ \nu \sqsubseteq \nu$.
4°. $\nu^{-1} \circ \nu^{-1} \sqsubseteq \nu$.

COROLLARY 2360. Every uniform space is symmetrically transitive.

PROPOSITION 2361. (Low)ν is a completely Cauchy space for every symmetrically transitive endoreloid ν.

PROOF. Suppose $S \in \mathscr{P}\left\{ \frac{\mathcal{X} \in \mathfrak{F}}{\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X} \sqsubseteq \nu} \right\}$.

$\bigsqcup\left\{ \frac{\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X}}{\mathcal{X} \in S} \right\} \sqsubseteq \nu$; $\bigsqcup\left\{ \frac{\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X}}{\mathcal{X} \in S} \right\} \circ \bigsqcup\left\{ \frac{\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X}}{\mathcal{X} \in S} \right\} \sqsubseteq \nu$; $\bigsqcup S \times^{\mathsf{RLD}} \bigsqcup S \sqsubseteq \nu$ (taken into account that S has nonempty meet). Thus $\bigsqcup S$ is Cauchy. □

PROPOSITION 2362. The neighbourhood filter $\langle(\mathsf{FCD})\nu\rangle^*\{x\}$ of a point $x \in \operatorname{Ob}\nu$ is a maximal Cauchy filter, if it is a Cauchy filter and ν is a reflexive reloid. FiXme: Does it holds for all low filters?

Proof. Let $\mathscr{N} = \langle(\mathsf{FCD})\nu\rangle^*\{x\}$. Let $\mathscr{C} \sqsupseteq \mathscr{N}$ be a Cauchy filter. We need to show $\mathscr{N} \sqsupseteq \mathscr{C}$.

Since $\mathscr{C}$ is Cauchy filter, $\mathscr{C} \times^{\mathsf{RLD}} \mathscr{C} \sqsubseteq \nu$. Since $\mathscr{C} \sqsupseteq \mathscr{N}$ we have $\mathscr{C}$ is a neighborhood of x and thus $\uparrow^{\mathrm{Ob}\,\nu}\{x\} \sqsubseteq \mathscr{C}$ (reflexivity of ν). Thus $\uparrow^{\mathrm{Ob}\,\nu}\{x\} \times^{\mathsf{RLD}} \mathscr{C} \sqsubseteq \mathscr{C} \times^{\mathsf{RLD}} \mathscr{C}$ and hence $\uparrow^{\mathrm{Ob}\,\nu}\{x\} \times^{\mathsf{RLD}} \mathscr{C} \sqsubseteq \nu$;

$$\mathscr{C} \sqsubseteq \operatorname{im}(\nu|_{\uparrow^{\mathrm{Ob}\,\nu}\{x\}}) = \langle(\mathsf{FCD})\nu\rangle^*\{x\} = \mathscr{N}.$$

□

10. Cauchy continuous functions

Definition 2363. A function $f : U \to V$ is *Cauchy continuous* from a low space $(U, \mathscr{C})$ to a low space $(V, \mathscr{D})$ when $\forall \mathcal{X} \in \mathscr{C} : \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{X} \in \mathscr{D}$.

Proposition 2364. Let f be a principal reloid. Then $f \in \mathrm{C}((\mathsf{RLD})_{\mathrm{Low}}\mathscr{C}, (\mathsf{RLD})_{\mathrm{Low}}\mathscr{D})$ iff f is Cauchy continuous.

$$\begin{aligned}
f \circ (\mathsf{RLD})_{\mathrm{Low}}\mathscr{C} \circ f^{-1} \sqsubseteq (\mathsf{RLD})_{\mathrm{Low}}\mathscr{D} \quad &\Leftrightarrow \\
\bigsqcup_{\mathcal{X} \in \mathscr{C}} (f \circ (\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X}) \circ f^{-1}) \sqsubseteq (\mathsf{RLD})_{\mathrm{Low}}\mathscr{D} \quad &\Leftrightarrow \\
\bigsqcup_{\mathcal{X} \in \mathscr{C}} (\langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{X} \times^{\mathsf{RLD}} \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{X}) \sqsubseteq (\mathsf{RLD})_{\mathrm{Low}}\mathscr{D} \quad &\Leftrightarrow \\
\forall \mathcal{X} \in \mathscr{C} : \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{X} \times^{\mathsf{RLD}} \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{X} \sqsubseteq (\mathsf{RLD})_{\mathrm{Low}}\mathscr{D} \quad &\Leftrightarrow \\
\forall \mathcal{X} \in \mathscr{C} : \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{X} \in \mathscr{D}. &
\end{aligned}$$

Thus we have expressed Cauchy properties through the algebra of reloids.

11. Cauchy-complete reloids

Definition 2365. An endoreloid ν is *Cauchy-complete* iff every low filter for this reloid converges to a point.

Remark 2366. In my book [2] *complete reloid* means something different. I will always prepend the word "Cauchy" to the word "complete" when meaning is by the last definition.

https://en.wikipedia.org/wiki/Complete_uniform_space#Completeness

12. Totally bounded

http://ncatlab.org/nlab/show/Cauchy+space

Definition 2367. Low space is called *totally bounded* when every proper filter contains a proper Cauchy filter.

Obvious 2368. A reloid ν is totally bounded iff

$$\forall X \in \mathscr{P}\,\mathrm{Ob}\,\nu \exists \mathcal{X} \in \mathfrak{F}^{\mathrm{Ob}\,\nu} : (\bot \neq \mathcal{X} \sqsubseteq \uparrow^{\mathrm{Ob}\,\nu} X \wedge \mathcal{X} \times^{\mathsf{RLD}} \mathcal{X} \sqsubseteq \nu).$$

Theorem 2369. A symmetric transitive reloid is totally bounded iff its Cauchy space is totally bounded.

Proof.

$\Rightarrow$. Let $\mathcal{F}$ be a proper filter on $\mathrm{Ob}\,\nu$ and let $a \in \operatorname{atoms}\mathcal{F}$. It's enough to prove that a is Cauchy.

Let $D \in \mathrm{GR}\,\nu$. Let also $E \in \mathrm{GR}\,\nu$ is symmetric and $E \circ E \subseteq D$. There existsa finite subset $F \subseteq \mathrm{Ob}\,\nu$ such that $\langle E\rangle F = \mathrm{Ob}\,\nu$. Then obviously exists $x \in F$ such that $a \sqsubseteq \uparrow^{\mathrm{Ob}\,\nu} \langle E\rangle\{x\}$, but $\langle E\rangle\{x\} \times \langle E\rangle\{x\} = E^{-1} \circ (\{x\} \times \{x\}) \circ E \subseteq D$, thus $a \times^{\mathsf{RLD}} a \sqsubseteq \uparrow^{\mathsf{RLD}(\mathrm{Ob}\,\nu, \mathrm{Ob}\,\nu)} D$.

Because D was taken arbitrary, we have $a \times^{\mathsf{RLD}} a \sqsubseteq \nu$ that is a is Cauchy.

$\Leftarrow$. Suppose that Cauchy space associated with a reloid ν is totally bounded but the reloid ν isn't totally bounded. So there exists a $D \in \operatorname{GR}\nu$ such that $(\operatorname{Ob}\nu) \setminus \langle D\rangle F \neq \emptyset$ for every finite set F.

Consider the filter base

$$S = \left\{ \frac{(\operatorname{Ob}\nu) \setminus \langle D\rangle F}{F \in \mathscr{P}\operatorname{Ob}\nu, F \text{ is finite}} \right\}$$

and the filter $\mathcal{F} = \bigsqcap \langle \uparrow^{\operatorname{Ob}\nu}\rangle S$ generated by this base. The filter $\mathcal{F}$ is proper because intersection $P \cap Q \in S$ for every $P, Q \in S$ and $\emptyset \notin S$. Thus there exists a Cauchy (for our Cauchy space) filter $\mathcal{X} \sqsubseteq \mathcal{F}$ that is $\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X} \sqsubseteq \nu$.

Thus there exists $M \in \mathcal{X}$ such that $M \times M \subseteq D$. Let F be a finite subset of $\operatorname{Ob}\nu$. Then $(\operatorname{Ob}\nu) \setminus \langle D\rangle F \in \mathcal{F} \sqsupseteq \mathcal{X}$. Thus $M \not\asymp (\operatorname{Ob}\nu) \setminus \langle D\rangle F$ and so there exists a point $x \in M \cap ((\operatorname{Ob}\nu) \setminus \langle D\rangle F)$.

$\langle M \times M\rangle\{p\} \subseteq \langle D\rangle\{x\}$ for every $p \in M$; thus $M \subseteq \langle D\rangle\{x\}$.

So $M \subseteq \langle D\rangle(F \cup \{x\})$. But this means that $M \in \mathcal{X}$ does not intersect $(\operatorname{Ob}\nu) \setminus \langle D\rangle(F \cup \{x\}) \in \mathcal{F} \sqsupseteq \mathcal{X}$, what is a contradiction (taken into account that $\mathcal{X}$ is proper).

□

http://math.stackexchange.com/questions/104696/pre-compactness-total-boundedness-and-cauchy-sequential-compactness

13. Totally bounded funcoids

Definition 2370. A funcoid ν is totally bounded iff

$$\forall X \in \operatorname{Ob}\nu \exists \mathcal{X} \in \mathfrak{F}^{\operatorname{Ob}\nu} : (0 \neq \mathcal{X} \sqsubseteq \uparrow^{\operatorname{Ob}\nu} X \wedge \mathcal{X} \times^{\mathsf{FCD}} \mathcal{X} \sqsubseteq \nu).$$

This can be rewritten in elementary terms (without using funcoidal product: $\mathcal{X} \times^{\mathsf{FCD}} \mathcal{X} \sqsubseteq \nu \Leftrightarrow \forall P \in \partial\mathcal{X} : \mathcal{X} \sqsubseteq \langle\nu\rangle P \Leftrightarrow \forall P \in \partial\mathcal{X}, Q \in \partial\mathcal{X} : P\ [\nu]^*\ Q \Leftrightarrow \forall P, Q \in \operatorname{Ob}\nu : (\forall E \in \mathcal{X} : (E \cap P \neq \emptyset \wedge E \cap Q \neq \emptyset) \Rightarrow P\ [\nu]^*\ Q)$.

Note that probably I am the first person which has written the above formula (for proximity spaces for instance) explicitly.

14. On principal low spaces

Definition 2371. A low space $(U, \mathscr{C})$ is *principal* when all filters in $\mathscr{C}$ are principal.

Proposition 2372. Having fixed a set U, principal reflexive low spaces on U bijectively correspond to principal reflexive symmetric endoreloids on U.

Proof. ?? http://math.stackexchange.com/questions/701684/union-of-cartesian-squares □

15. Rest

https://en.wikipedia.org/wiki/Cauchy_filter#Cauchy_filters

https://en.wikipedia.org/wiki/Uniform_space "Hausdorff completion of a uniform space" here)

http://at.yorku.ca/z/a/a/b/13.htm : the category **Prox** of proximity spaces and proximally continuous maps (i.e. maps preserving nearness between two sets) is isomorphic to the category of totally bounded uniform spaces (and uniformly continuous maps).

https://en.wikipedia.org/wiki/Cauchy_space http://ncatlab.org/nlab/show/Cauchy+space

http://arxiv.org/abs/1309.1748

http://projecteuclid.org/download/pdf_1/euclid.pja/1195521991

http://www.emis.de/journals/HOA/IJMMS/Volume5_3/404620.pdf

~/math/books/Cauchy_spaces.pdf

https://ncatlab.org/nlab/show/Cauchy+space defines compact Cauchy spaces!

http://www.hindawi.com/journals/ijmms/1982/404620/abs/ (open access article) describes criteria for a Cauchy space to be uniformizable.

CHAPTER 12

Funcoidal groups

REMARK 2373. FiXme: Move this into the book. If μ and ν are cocomplete endofuncoids, then we can describe $f \in \mathrm{C}(\mu, \nu)$ without using filters by the formulas:

1°. $\langle f \rangle^* \langle \mu \rangle^* X \sqsubseteq \langle \nu \rangle^* \langle f \rangle^* X$ (for every set X in $\mathscr{P} \operatorname{Ob} \mu$)
2°. $\langle \mu \rangle^* X \sqsubseteq \langle f^{-1} \rangle^* \langle \nu \rangle^* \langle f \rangle^* X$ (for every set X in $\mathscr{P} \operatorname{Ob} \mu$)
3°. $\langle f \rangle^* \langle \mu \rangle^* \langle f^{-1} \rangle^* Y \sqsubseteq \langle \nu \rangle^* Y$ (for every set Y in $\mathscr{P} \operatorname{Ob} \nu$)

Funcoidal groups are modeled after topological groups (see Wikipedia) and are their generalization.

DEFINITION 2374. *Funcoidal group* is a group G together with endofuncoid μ on $\operatorname{Ob} G$ such that

1°. $(y\cdot) \in \mathrm{C}(\mu; \mu)$ for every $y \in G$;
2°. $(\cdot x) \in \mathrm{C}(\mu; \mu)$ for every $x \in G$;
3°. $(x \mapsto x^{-1}) \in \mathrm{C}(\mu; \mu)$ for every $x \in G$.

PROPOSITION 2375. $t \mapsto y \cdot t \cdot x$ and $t \mapsto y \cdot t^{-1} \cdot x$ are continuous functions.

PROOF. As composition of continuous functions. □

OBVIOUS 2376. Composition of functions of the forms $t \mapsto y \cdot t \cdot x$ and $t \mapsto y \cdot t^{-1} \cdot x$ are also a function of one of these forms.

What is the purpose of the following (yet unproved) proposition? I don't know, but it looks curious.

PROPOSITION 2377. Let E be a composition of functions of a form $\langle \mu \rangle^*$, $\langle y\cdot \rangle^*$, $\langle \cdot x \rangle^*$, $\langle ^{-1} \rangle^*$ (where x and y vary arbitrarily) such that μ is met in the composition at least once. Let also either $\mu = \mu \circ \mu$ or μ is met exactly once in the product. There are such elements x_0, y_0 that either

1°. $(t \mapsto y_0 \cdot t \cdot x_0) \circ \langle \mu \rangle \sqsubseteq E \sqsubseteq \langle \mu \rangle \circ (t \mapsto y_0 \cdot t \cdot x_0)$;
2°. $(t \mapsto y_0 \cdot t^{-1} \cdot x_0) \circ \langle \mu \rangle \sqsubseteq E \sqsubseteq \langle \mu \rangle \circ (t \mapsto y_0 \cdot t^{-1} \cdot x_0)$.

PROOF. Using continuity a few times we prove that $E \sqsubseteq \langle \mu \rangle^* \circ \ldots \circ \langle \mu \rangle^* \circ f_n \circ \ldots \circ f_1$ where f_i are functions of the forms $t \mapsto y \cdot t \cdot x$ or $t \mapsto y \cdot t^{-1} \cdot x$ for $n \in \mathbb{N}$. But $\langle \mu \rangle^* \circ \ldots \circ \langle \mu \rangle^* = \langle \mu \rangle^*$ by conditions and $f_n \circ \ldots \circ f_1$ is of the form $t \mapsto y \cdot t \cdot x$ or $t \mapsto y \cdot t^{-1} \cdot x$ by above proposition. $E \sqsubseteq \langle \mu \rangle \circ (t \mapsto y_0 \cdot t \cdot x_0)$ or $E \sqsubseteq \langle \mu \rangle \circ (t \mapsto y_0 \cdot t^{-1} \cdot x_0)$

The second inequalty is similar. Note that x_0 and y_0 are the same for the first and for the second item. □

(G, μ) vs (G, μ^{-1}) are they isomorphic?
FiXme: We can also define reloidal groups.

1. On “Each regular paratopological group is completely regular” article

In this chapter I attempt to rewrite the paper [1] in more general setting of funcoids and reloids. I attempt to construct a “royal road” to finding proofs of statements of this paper and similar ones, what is important because we lose 60 years waiting for any proof.

1.1. Definition of normality. By definition (slightly generalizing the special case if μ is a quasi-uniform space from [1]) a pair of an endo-reloid μ and a complete funcoid ν (playing role of a generalization of a topological space) on a set U is *normal* when

$$\left\langle \nu^{-1} \right\rangle^{*} A \sqsubseteq \left\langle \nu^{-1\circ} \right\rangle^{*} \left\langle \nu^{-1} \right\rangle^{*} \langle F \rangle^{*} A$$

for every entourage $F \in \operatorname{up} \mu$ of μ and every set $A \subseteq U$.

Note that this is *not* the same as customary definition of normal topological spaces.

THEOREM 2378. An endoreloid μ is normal on endoreloid ν iff

$$\nu \circ \nu^{-1} \sqsubseteq \nu^{-1} \circ (\mathsf{FCD})\mu.$$

PROOF. Equivalently transforming the criterion of normality (which should hold for all $F \in \operatorname{up} \mu$) using proposition 2222:

$\langle \nu \rangle^{*} \left\langle \nu^{-1} \right\rangle^{*} A \sqsubseteq \left\langle \nu^{-1} \right\rangle^{*} \langle F \rangle^{*} A$.

Also note $\prod^{\mathscr{F}}_{F \in \operatorname{up} \mu} \left\langle \nu^{-1} \right\rangle^{*} \langle F \rangle^{*} A =$ (because funcoids preserve filtered meets) $= \left\langle \nu^{-1} \right\rangle^{*} \prod^{\mathscr{F}}_{F \in \operatorname{up} \mu} \langle F \rangle^{*} A = \left\langle \nu^{-1} \right\rangle^{*} \langle (\mathsf{FCD})\mu \rangle^{*} A$.

Thus the above is equivalent to $\langle \nu \rangle^{*} \left\langle \nu^{-1} \right\rangle^{*} A \sqsubseteq \left\langle \nu^{-1} \right\rangle^{*} \langle (\mathsf{FCD})\mu \rangle^{*} A$.

And this is in turn equivalent to

$$\nu \circ \nu^{-1} \sqsubseteq \nu^{-1} \circ (\mathsf{FCD})\mu.$$

□

DEFINITION 2379. An endofuncoid μ is *normal* on endofuncoid ν when $\nu \circ \nu^{-1} \sqsubseteq \nu^{-1} \circ \mu$. FiXme: No need for ν to be endomorphism.

OBVIOUS 2380.

1°. Endoreloid μ is normal on endofuncoid ν iff endofuncoid $(\mathsf{FCD})\mu$ is normal on endofuncoid ν.

2°. Endofuncoid μ is normal on endoreloid ν iff endofuncoid $(\mathsf{RLD})_{\mathrm{in}}\mu$ is normal on endofuncoid ν.

COROLLARY 2381. If ν is a symmetric endofuncoid and $\mu \sqsupseteq \nu^{-1}$, then it is normal.

COROLLARY 2382. (generalization of proposition 1 in [1]) If ν is a symmetric endofuncoid and $\operatorname{Compl} \mu \sqsupseteq \nu^{-1}$, then it is normal.

DEFINITION 2383. A funcoid ν is *normally reloidazable* iff there exist a reloid μ such that (μ, ν) is normal and $\nu = \operatorname{Compl}(\mathsf{FCD})\mu$.

DEFINITION 2384. A funcoid ν is *normally quasi-uniformizable* iff there exist a quasi-uniform space ($=$ reflexive and transitive reloid) μ such that (μ, ν) is normal and $\nu = \operatorname{Compl}(\mathsf{FCD})\mu$.

PROPOSITION 2385. A funcoid ν is normally reloidazable iff there exist a funcoid μ such that μ is normal on ν and $\nu = \operatorname{Compl} \mu$.

PROPOSITION 2386. A funcoid ν is normally quasi-uniformizable iff there exist a quasi-proximity space (= reflexive and transitive funcoid) μ such that μ is normal on ν and $\nu = \operatorname{Compl}\mu$.

PROOF. Obvious 2380 and the fact that (FCD) is an isomorphism between reflexive and transitive funcoids and reflexive and transitive reloids. $\square$

In other words, it is normally reloidazable or normally quasi-uniformizable when

$$(\operatorname{Compl}\mu)\circ(\operatorname{Compl}\mu)^{-1}\sqsubseteq(\operatorname{Compl}\mu)^{-1}\circ\mu$$

for suitable μ.

1.2. Urysohn's lemma and friends. For a detailed proof of Urysohn's lemma see also:
http://homepage.math.uiowa.edu/~jsimon/COURSES/M132Fall07/UrysohnLemma_v5.pdf
https://proofwiki.org/wiki/Urysohn's_Lemma
http://planetmath.org/proofofurysohnslemma

https://en.wikipedia.org/wiki/Proximity_space says that “The resulting topology is always completely regular. This can be proven by imitating the usual proofs of Urysohn's lemma, using the last property of proximal neighborhoods to create the infinite increasing chain used in proving the lemma.”

Below follows an alternative proof of Urysohn lemma. *The proof was based on a conjecture proved false, see example 1444!*

LEMMA 2387. If $\langle\mu\rangle\mathcal{A}\asymp\mathcal{B}$ for a complete funcoid μ and $\mathcal{A}$, $\mathcal{B}$ are filters on relevant sets, then there exists $U\in\operatorname{up}\mu$ such that $\langle U\rangle\mathcal{A}\asymp\mathcal{B}$.

PROOF. Prove that $\left\{\frac{\langle U\rangle\mathcal{A}}{U\in\operatorname{up}\mu}\right\}$ is a filter base. That it is nonempty is obvious.

Let $\mathcal{X},\mathcal{Y}\in\left\{\frac{\langle U\rangle\mathcal{A}}{U\in\operatorname{up}\mu}\right\}$. Then $\mathcal{X}=\langle U_{\mathcal{X}}\rangle\mathcal{A}$, $Y=\langle U_{\mathcal{Y}}\rangle\langle A\rangle$. Because μ is complete, we have (proposition 1227) $U_{\mathcal{X}}\sqcap U_{\mathcal{Y}}\in\operatorname{up}\mu$. Thus $\mathcal{X},\mathcal{Y}\sqsupseteq\langle U_{\mathcal{X}}\sqcap U_{\mathcal{Y}}\rangle\mathcal{A}\in\left\{\frac{\langle U\rangle\mathcal{A}}{U\in\operatorname{up}\mu}\right\}$.

Thus $\langle\mu\rangle\mathcal{A}\asymp\mathcal{B}\Leftrightarrow\mathcal{B}\sqcap\langle\mu\rangle\mathcal{A}=\bot\Leftrightarrow\exists U\in\operatorname{up}\mu:\mathcal{B}\sqcap\langle U\rangle\mathcal{A}=\bot\Leftrightarrow\exists U\in\operatorname{up}\mu:\langle U\rangle\mathcal{A}\asymp\mathcal{B}$. $\square$

COROLLARY 2388. If $\langle\mu\rangle\mathcal{A}\asymp\langle\mu\rangle\mathcal{B}$ for a complete funcoid μ and $\mathcal{A}$, $\mathcal{B}$ are filters on relevant sets, then there exists $U\in\operatorname{up}\mu$ such that $\langle U\rangle\mathcal{A}\asymp\langle U\rangle\mathcal{B}$.

PROOF. Applying the lemma twice we can obtain $P,Q\in\operatorname{up}\mu$ such that $\langle P\rangle\mathcal{A}\asymp\langle Q\rangle\mathcal{B}$. But because μ is complete, we have $U=P\sqcap Q\in\operatorname{up}\mu$, while obviously $\langle U\rangle\mathcal{A}\asymp\langle U\rangle\mathcal{B}$. $\square$

LEMMA 2389. (assuming conjecture 1444) For every $U\in\operatorname{up}\mu$ (where μ is a T_4 topological space) such that $\neg\left(A\left[U\circ U^{-1}\right]^{*}B\right)$ there is $W\in\operatorname{up}\mu$ such that $U\circ U^{-1}\sqsupseteq W\circ W^{-1}\circ W\circ W^{-1}$. For it holds $\neg\left(A\left[W\circ W^{-1}\right]^{*}B\right)$. We can assume that $\langle W\rangle^{*}X$ is open for every set X.

PROOF. $U\circ U^{-1}\in\operatorname{up}(\mu\circ\mu^{-1})\subseteq\operatorname{up}(\mu\circ\mu^{-1}\circ\mu\circ\mu^{-1})$ (normality used). Thus by the conjecture there exists $W\in\operatorname{up}\mu$ such that $U\circ U^{-1}\sqsupseteq W\circ W^{-1}\circ W\circ W^{-1}$. $W\circ W^{-1}\sqsubseteq U\circ U^{-1}$ thus $\neg\left(A\left[W\circ W^{-1}\right]^{*}B\right)$.

To prove that $\langle W\rangle^{*}X$ is open for every set X, replace every $\langle W\rangle^{*}\{x\}$ with an open neighborhood $E\subseteq\langle W\rangle^{*}X$ of $\langle\mu\rangle^{*}\{x\}$ (and note that union of open sets is open). This new W holds all necessary properties. $\square$

LEMMA 2390. (assuming conjecture 1444) For every $U \in \operatorname{up} \mu$ (where μ is a T_4 topological space) such that $\neg\left(A \left[U \circ U^{-1}\right]^* B\right)$ there is $W \in \operatorname{up} \mu$ such that $U \circ U^{-1} \sqsupseteq \mu^{-1} \circ W \circ W^{-1} \circ W \circ W^{-1}$. For it holds $\neg\left(A \left[W \circ W^{-1}\right]^* B\right)$. We can assume that $\langle W \rangle^* X$ is open for every set X.

PROOF. Applying the previous lemma twice, we have some open $W \in \operatorname{up} \mu$ such that

$$U \circ U^{-1} \sqsupseteq W \circ W^{-1} \circ W \circ W^{-1} \circ W \circ W^{-1} \circ W \circ W^{-1}$$

and $\neg\left(A \left[W \circ W^{-1}\right]^* B\right)$. From this easily follows that

$$U \circ U^{-1} \sqsupseteq \mu^{-1} \circ W \circ W^{-1} \circ W \circ W^{-1}.$$

□

A modified proof of Urysohn's lemma follows. This proof is in part based on [1]. (I attempt to find common generalization of Urysohn's lemma and results from [1]).

$\mathbb{Q}_2 \stackrel{\text{def}}{=} \left\{ \frac{k/2^n}{k,n \in \mathbb{N}, 0<k<2^n} \right\}$.

THEOREM 2391. Urysohn's lemma (see Wikipedia) for disjoint closed sets A and B and function f on a topological space μ (considered as complete funcoid).

PROOF. (assuming conjecture 1444) (used ProofWiki among other sources)

Because A and B are disjoint closed sets, we have $\langle \mu \rangle^* A \asymp \langle \mu \rangle^* B$. Thus by the corollary 2388 take $S_0 \in \operatorname{up} \mu$ and $\neg\left(A \left[S_0 \circ S_0^{-1}\right]^* B\right)$.

We have $\mu \circ \mu^{-1} \circ \mu \circ \mu^{-1} \sqsubseteq \mu \circ \mu^{-1}$ that is $\operatorname{up}(\mu \circ \mu^{-1} \circ \mu \circ \mu^{-1}) \supseteq \operatorname{up}(\mu \circ \mu^{-1})$.

Let's prove by induction: There is a sequence S of binary relations starting with S_0 such that $\neg\left(A \left[S_i \circ S_i^{-1}\right]^* B\right)$ and $S_i \circ S_i^{-1} \sqsupseteq \mu^{-1} \circ S_{i+1} \circ S_{i+1}^{-1} \circ S_{i+1} \circ S_{i+1}^{-1}$. It directly follows from the lemma (and uses the conjecture).

Denote $U_i = S_{i+1} \circ S_{i+1}^{-1}$. We have $U_i \sqsupseteq \mu^{-1} \circ U_{i+1} \circ U_{i+1}$ and $\neg\left(A \left[U_i\right]^* B\right)$.

By reflexivity of μ we have $U_{i+1} \subseteq U_{i+1} \circ U_{i+1} \subseteq U_i$.

Define fractional degree of U: $U^r \stackrel{\text{def}}{=} U_1^{r_1} \circ \ldots \circ U_{l_r}^{r_{l_r}}$ for every $r \in \mathbb{Q}_2$ where $r_1 \ldots r_{l_r}$ is the binary expansion of r.

Prove $U_r \subseteq U_0$. It is enough to prove $U_0 \supseteq U_1 \circ \ldots \circ U_{l_r}$. It follows from $U_2 \circ \ldots \circ U_{l_r} \subseteq U_1$, $U_3 \circ \ldots \circ U_{l_r} \subseteq U_2$, …, $U_{l_r} \subseteq U_{l_r - 1}$ what was shown above.

Let's prove: For each $p, q \in \mathbb{Q}_2$ such that $p < q$ we have $\mu^{-1} \circ U^p \sqsubseteq U^q$. We can assume binary expansion of p and q be the same length c (add zeros at the end of the shorter one). Now it is enough to prove

$$U_k \circ U_{k+1}^{q_{k+1}} \circ \cdots \circ U_c^{q_c} \sqsupseteq \mu^{-1} \circ U_{k+1}^{p_{k+1}} \circ U_{k+2}^{p_{k+2}} \circ \cdots \circ U_c^{p_c}.$$

But for this it's enough

$$U_k \sqsupseteq \mu^{-1} \circ U_{k+1} \circ U_{k+2} \circ \cdots \circ U_c$$

what can be easily proved by induction: If $k = c$ then it takes the form $U_k \sqsupseteq \mu^{-1}$ what is obvious. Suppose it holds for k. Then $U_{k-1} \sqsupseteq \mu^{-1} \circ U_k \circ U_k \sqsupseteq \mu^{-1} \circ U_k \circ \mu^{-1} \circ U_{k+1} \circ U_{k+2} \circ \cdots \circ U_c \sqsupseteq \mu^{-1} \circ U_k \circ U_{k+1} \circ U_{k+2} \circ \cdots \circ U_c$, that is it holds for all natural $k \leq c$.

It is easy to prove that $\langle U^r \rangle^* X$ is open for every set X.

We have $\left\langle \mu^{-1} \right\rangle^* \langle U^p \rangle^* X \sqsubseteq \langle U^q \rangle^* X$.

$$f(z) \stackrel{\text{def}}{=} \inf\left(\{1\} \cup \left\{ \frac{q \in \mathbb{Q}_2}{z \in \langle U^q \rangle^* A} \right\}\right).$$

f is properly defined because $\{1\} \cup \left\{ \frac{q \in \mathbb{Q}_2}{z \in \langle U^q \rangle^* A} \right\}$ is nonempty and bounded.

If $z \in A$ then $z \in \langle U^q \rangle^* A$ for every $q \in \mathbb{Q}_2$, thus $f(z) = 0$, because obviously $U^q \sqsupseteq 1$.

If $z \in B$ then $z \notin \langle U^q \rangle^* A$ for every $q \in \mathbb{Q}_2$, thus $f(z) = 1$, because $U^q \sqsubseteq U_0$.

It remains to prove that f is continuous.

Let $D(x) = \{1\} \cup \left\{ \frac{q \in \mathbb{Q}_2}{z \in \langle U^q \rangle^* A} \right\}$.

To show that f is continuous, we first prove two smaller results:

(a) $x \in \langle \mu^{-1} \rangle^* \langle U^r \rangle^* A \Rightarrow f(x) \leq r$.

We have $x \in \langle \mu^{-1} \rangle^* \langle U^r \rangle^* A \Rightarrow \forall s > r : x \in \langle U^s \rangle^* A$, so $D(x)$ contains all rationals greater than r. Thus $f(x) \leq r$ by definition of f.

(b) $x \notin \langle U^r \rangle^* A \Rightarrow f(x) \geq r$.

We have $x \notin \langle U^r \rangle^* A \Rightarrow \forall s < r : x \notin \langle U^s \rangle^* A$. So $D(x)$ contains no rational less than r. Thus $f(x) \geq r$.

Let $x_0 \in S$ and let $]c; d[$ be an open real interval containing $f(x)$. We will find a neighborhood T of x_0 such that $\langle f \rangle^* T \subseteq]c; d[$.

Choose $p, q \in \mathbb{Q}$ such that $c < p < f(x_0) < q < d$. Let $T = \langle U^q \rangle^* A \setminus \langle \mu^{-1} \rangle^* \langle U^p \rangle^* A$.

Then since $f(x_0) < q$, we have that (b) implies vacuously that $x \in \langle U^q \rangle^* A$.

Since $f(x_0) > p$, (a) implies $x_0 \notin \langle U^p \rangle^* A$.

Hence $x_0 \in T$. Then T is a neighborhood of x_0 because T is open.

Finally, let $x \in T$.

Then $x \in \langle U^q \rangle^* A \subseteq \langle \mu^{-1} \rangle^* \langle U^q \rangle^* A$. So $f(x) \leq q$ by (a).

Also $x \notin \langle \mu^{-1} \rangle^* \langle U^p \rangle^* A$, so $x \notin \langle U^p \rangle^* A$ and $f(x) \geq p$ by (b).

Thus: $f(x) \in [p; q] \subseteq]c; d[$.

Therefore f is continuous.

Claim A: $f(x) > q \Rightarrow x \notin \langle \mu^{-1} \rangle^* \langle U^q \rangle^* A$

Claim B: $f(x) < q \Rightarrow x \in \langle U^q \rangle^* A$

Proof of claim A: If $f(x) > q$ then then there must be some gap between q and $D(x)$; in particular, there exists some q' such that $q < q' < f(x)$. But $q' < f(x) \Rightarrow x \notin \langle U^q \rangle^* A \Rightarrow x \notin \langle \mu^{-1} \rangle^* \langle U^q \rangle^* A$ (using that $\langle U^r \rangle^* X$ is open).

Proof of claim B: If $f(x) < q$ then there exists $q' \in D(x)$ such that $f(x) < q' < q$, in which case $q \in D(x)$, so $x \in \langle U^q \rangle^* A$.

To show that f is continuous, it's enough to prove that preimages of $]a; 1]$ and $[0; a[$ are open.

Suppose $f(x) \in]a; 1]$. Pick some q with $a < q < f(x)$. We claim that the open set $W = X \setminus \langle f^{-1} \rangle^* \langle U^q \rangle^* A$ is a neighborhood of x that is mapped by f into $]a; 1]$. First, by (A), $f(x) > q \Rightarrow x \in W$, so W is a neighborhood of x. If y is any point of W, then $f(y)$ must be $\geq q > a$; otherwise, if $f(y) < q$, then, by (B) $y \in \langle U^q \rangle^* A \subseteq \langle f^{-1} \rangle^* \langle U^q \rangle^* A$.

Suppose $x \in f^{-1}[0; b[$ that is $f(x) < b$ and pick q such that $f(x) < q < b$. By (B) $x \in \langle U^q \rangle^* A$. We claim that the neighborhood $\langle U^q \rangle^* A$ is mapped by f into $[0; b[$. Suppose y is any point of $\langle U^q \rangle^* A$. Then $q \in D(y)$, so $f(y) \leq q < b$. □

Theorem 2392. (from [1]) If μ is a normal quasi-uniformity on a topological space ν, then for any nonempty subset $A \in \operatorname{Ob} \nu$ and entourage $U \in \operatorname{up} \mu$ there exists a continuous function $f : \operatorname{Ob} \nu \to [0; 1]$ such that $A \sqsubseteq \langle f^{-1} \rangle^* \{0\} \sqsubseteq \langle f^{-1} \rangle^* [0; 1[\sqsubseteq \langle \nu^{-1\circ} \rangle^* \langle \nu^{-1} \rangle^* \langle U \rangle^* A$.

Proof. Choose inductively a sequence of entourages $(U_n)_{n=0}^{\infty}$ such that $U_0 = U$ and $U_{n+1} \circ U_{n+1} \sqsubseteq U_n$.

Denote $l_r = \max \left\{ \frac{n \in \mathbb{N}}{r_n = 1} \right\}$.

Define $U^r = U_{l_r}^{r_{l_r}} \circ \ldots \circ U_1^{r_1}$

Prove $\langle\nu^{-1}\rangle^*\langle U^q\rangle^* A \sqsubseteq \langle\nu^{-1}\circ\rangle^*\langle\nu^{-1}\rangle^*\langle U^r\rangle^* A$ for any $q < r$ in $\mathbb{Q}_2$. FiXme: Can be easily rewritten with the formula $\langle\nu\rangle^*\langle\nu^{-1}\rangle^*\langle U^q\rangle^* A \sqsubseteq \langle\nu^{-1}\rangle^*\langle U^r\rangle^* A$ instead. It may extend to non-complete funcoids.

There is such l that $0 = q_l < r_l = 1$ and $q_i = r_i$ for all $i < l$.

It follows $l_q \neq l \leq l_r$.

Consider variants:

$l_q < l$. $\langle\nu^{-1}\rangle^*\langle U^q\rangle^* A \sqsubseteq \langle\nu^{-1}\rangle^*\left\langle U_{l_q}\circ\ldots\circ U_1^{q_1 q_{l_q}}\right\rangle^* A = \langle\nu^{-1}\rangle^*\left\langle U_{l_q}^{r_{l_q}}\circ\ldots\circ U_1^{r_1}\right\rangle^* A \sqsubseteq \langle\nu^{-1}\rangle^*\langle U_{l-1}^{r_{l-1}}\circ\ldots\circ U_1^{r_1}\rangle^* A \sqsubseteq \langle\nu^{-1}\circ\rangle^*\langle\nu^{-1}\rangle^*\langle U_l^{r_l}\circ U_{l-1}^{r_{l-1}}\circ\ldots\circ U_1^{r_1}\rangle^* A = \langle\nu^{-1}\circ\rangle^*\langle\nu^{-1}\rangle^*\langle U^r\rangle^* A$ (use $U_l^{r_l}\in\operatorname{up}(\mathsf{FCD})\mu$ by theorem 992).

$l < l_q$. Inclusions $U_k\circ U_k \sqsubseteq U_{k-1}$ for $l < k \leq l_q+1$ guarantee that $U_{l_q+1}\circ U_{l_q}\circ\ldots\circ U_{l+1}\sqsubseteq U_l$ and then $\langle\nu^{-1}\rangle^*\langle U^q\rangle^* A \sqsubseteq \langle\nu^{-1}\rangle^*\left\langle U_{l_q}^{q_{l_q}}\circ\ldots\circ U_1^{q_1}\right\rangle^* A \sqsubseteq \langle\nu^{-1}\circ\rangle^*\langle\nu^{-1}\rangle^*\left\langle U_{l_q+1}^{q_{l_q+1}}\circ U_{l_q}^{q_{l_q}}\circ\ldots\circ U_1^{q_1}\right\rangle^* A = \langle\nu^{-1}\circ\rangle^*\langle\nu^{-1}\rangle^*\left\langle U_{l_q+1}\circ U_{l_q}^{q_{l_q}}\circ\ldots\circ U_l^0\circ\ldots\circ U_1^{q_1}\right\rangle^* A \sqsubseteq \langle\nu^{-1}\circ\rangle^*\langle\nu^{-1}\rangle^*\langle U_l\circ U_{l-1}^{q_{l-1}}\circ\ldots\circ U_1^{q_1}\rangle^* A \sqsubseteq \langle\nu^{-1}\circ\rangle^*\langle\nu^{-1}\rangle^*\langle U_l^{r_l}\circ U_{l-1}^{r_{l-1}}\circ\ldots\circ U_1^{r_1}\rangle^* A \sqsubseteq \langle\nu^{-1}\circ\rangle^*\langle\nu^{-1}\rangle^*\langle U_{l_r}^{r_{l_r}}\circ\ldots\circ U_1^{r_1}\rangle^* A = \langle\nu^{-1}\circ\rangle^*\langle\nu^{-1}\rangle^*\langle U_r\rangle^* A$.

Define f by the formula $f(z) = \inf\left(\{1\}\cup\left\{\frac{q\in\mathbb{Q}_2}{z\in\langle\nu^{-1}\rangle^*\langle U^q\rangle^* A}\right\}\right)$.

It is clear?? that $A \sqsubseteq \langle f^{-1}\rangle^*\{0\}$ and $\langle f^{-1}\rangle^*[0;1[\sqsubseteq \bigcup_{q\in\mathbb{Q}_2}\langle\nu^{-1}\rangle^*\langle U^q\rangle^* A = \bigcup_{r\in\mathbb{Q}_2}\langle\nu^{-1}\circ\rangle^*\langle\nu^{-1}\rangle^*\langle U^r\rangle^* A \sqsubseteq \langle\nu^{-1}\circ\rangle^*\langle\nu^{-1}\rangle^*\langle U_0\rangle^* A$.

To prove that the map $f : X \to [0,1]$ is continuous, it suffices to check that for every real number $a\in]0;1[$ the sets $\langle f^{-1}\rangle^*[0;a[$ and $\langle f^{-1}\rangle^*]a;1]$ are open. This follows from the equalitites

$\langle f^{-1}\rangle^*[0;a[=\bigcup_{\mathbb{Q}_2\ni q<a}\langle\nu^{-1}\circ\rangle^*\langle\nu^{-1}\rangle^*\langle U^q\rangle^* A$ and $\langle f^{-1}\rangle^*]a;1] = \bigcup_{\mathbb{Q}_2\ni r>a}(X\setminus\langle\nu^{-1}\rangle^*\langle U^r\rangle^* A)$. □

How the formulas for normal (T_4) topological spaces and normal quasi-uniformities are related? Maybe this works: Replacing $\nu\to\mu\circ\mu^{-1}$, $\mu\to 1$ makes $\nu\circ\nu^{-1}\sqsubseteq\nu^{-1}\circ(\mathsf{FCD})\mu \to \mu\circ\mu^{-1}\circ\mu\circ\mu^{-1}\sqsubseteq\mu\circ\mu^{-1}$.

https://www.researchgate.net/project/The-lattice-LG-of-group-topologies

CHAPTER 13

Micronization

I defined "micronization" wrongly in my book and did some erroneous proofs about it. Here is an attempt to salvage it.

https://en.wikipedia.org/wiki/Transitive_reduction is a special case of micronization. (Hm, maybe them coincide only for finite sets?)

DEFINITION 2393. *Micronization* $\mu(E)$ of a binary relation E is defined by the formula:

$$\mu(E) = \prod^{\mathsf{RLD}} \left\{ \frac{f \in \mathsf{RLD}}{S^*(f) \sqsupseteq E \wedge f \asymp f^2.} \right\}$$

It's wrong (consider micronization of $\leq$ on real numbers (which should be addition of infinite small).

QUESTION 2394. Under which conditions $S^*(\mu(E)) = E$?

CHAPTER 14

More on connectedness

1. For topological spaces

PROPOSITION 2395. The following are pairwise equivalent:

1°. a topological space on a set U is connected. FiXme: definition; can the topological definition be generalized to filters?
2°. U is connected regarding $f \sqcup f^{-1}$ if f is the corresponding complete funcoid.
3°. U is connected regarding $f \sqcup f^{-1}$ if f is the corresponding closure space.
4°. U is connected regarding $f \circ f^{-1}$ if f is the corresponding complete funcoid.

PROOF. ?? □

PROPOSITION 2396. There are filters $\mathcal{A}$, $\mathcal{B}$, such that there are no filters $\mathcal{X} \sqsubseteq \mathcal{A}$, $\mathcal{Y} \sqsubseteq \mathcal{B}$ such that $\mathcal{X} \sqcup \mathcal{Y} = \mathcal{A} \sqcup \mathcal{B}$ and $\mathcal{X} \asymp \mathcal{Y}$.

PROOF. https://math.stackexchange.com/questions/2639206
(It also follows that sometimes $Z(Da)$ is not a complete lattice, because otherwise we could prove this theorem.) □

PROPOSITION 2397. If $\mathcal{A}$, $\mathcal{B}$ are filters and $\mathcal{A} \sqcup \mathcal{B} = U$ is principal filter, then there are sets $X \sqsubseteq \mathcal{A}$, $Y \sqsubseteq \mathcal{B}$ such that $X \sqcup Y = U$ and $X \asymp Y$.

PROOF. Take $X = \operatorname{Cor} \mathcal{A}$ and $Y' = \operatorname{Cor} \mathcal{B}$. Then $X \sqcup Y' = U$ because of co-separability of $\mathfrak{F}(U)$. Take $Y = U \setminus X$. Then $X \sqcup Y = U$ and $X \asymp Y$. □

PROPOSITION 2398. A principal filter A is connected regarding endofuncoid μ iff

$$\forall X, Y \in \mathscr{P}(\operatorname{Ob} \mu) \setminus \{\bot\} : (X \sqcup Y = A \wedge X \asymp Y \Rightarrow X \; [\mu] \; Y).$$

PROOF. Easily follows from ??. □

DEFINITION 2399. *Connected component* of a filter regarding a funcoid or a reloid is a maximal connected subfilter of this filter.

OBVIOUS 2400. Subfilter of a connected filter is connected.

PROPOSITION 2401. If U is a principal filter, then it is connected regarding μ iff it is connected regarding $S(\mu)$. FiXme: It should be presented as a corollary of a below theorem.

PROOF. If U is connected regarding μ, it is connected regarding $S(\mu)$, obviously.

Suppose U is connected regarding $S(\mu)$. Then for $X, Y \in \mathscr{P}(\operatorname{Ob} \mu) \setminus \{\bot\}$ we have if $X \sqcup Y = U$ and $X \asymp Y$, then $X \; [S(\mu)] \; Y$. So $X \times Y \not\asymp 1 \sqcup \mu \sqcup \mu^2 \sqcup \ldots$ and thus by distributivity for principal filter we have $X \times Y \not\asymp \mu^n$ for some $n \geq$?? that is $X \; [\mu^n] \; Y$ for some n and thus there are atomic filters $p_0, \ldots, p_n$ such that $p_0 \in \operatorname{atoms}^{\mathfrak{F}} X$, $p_n \in \operatorname{atoms}^{\mathfrak{F}} Y$ and $p_i \; [\mu] \; p_{i+1}$. Thus there is k such that $p_k \; [\mu] \; p_{k+1}$ and $p_k \in \operatorname{atoms}^{\mathfrak{F}} X$, $p_{k+1} \in \operatorname{atoms}^{\mathfrak{F}} Y$. Thus $X \; [\mu] \; Y$. We have U connected regarding μ. □

Also for S^*

EXAMPLE 2402. Connected components may not form a weak partition.

PROOF. Consider funcoid $1^{\mathsf{FCD}(\mathbb{R})} \sqcup (\Delta \times^{\mathsf{FCD}} \Delta)$ on real line. Then connected components are (prove!) non-zero singletons and Δ. It is not a weak partition. $\square$

CONJECTURE 2403. If the set of connected components is finite, then it is a strong partition. Moreover the set of connected components is a tearing.

Add more counter-examples (for non-principal filters).

OBVIOUS 2404. Improper filter $\bot^{\mathscr{F}}$ is connected regarding:

1°. every funcoid;
2°. every reloid.

PROPOSITION 2405. The only filter connected regarding

1°. $\bot^{\mathsf{FCD}(A)}$;
2°. $\bot^{\mathsf{RLD}(A)}$

is the improper filter $\bot^{\mathscr{F}}$.

PROOF.

1°. Let $\mathcal{A}$ be a filter. Take $\mathcal{X} = \mathcal{Y} = \mathcal{A} \in \mathscr{F}(\operatorname{Ob}\mu) \setminus \{\bot\}$. Then $\mathcal{X} \sqcup \mathcal{Y} = \mathcal{A}$ but not $\mathcal{X} \; [\mu] \; \mathcal{Y}$.
2°. $S_1^*(\bot^{\mathsf{RLD}(A)}) = S_1(\bot^{\mathsf{RLD}(A)}) = \bot^{\mathsf{RLD}(A)}$. Thus the only connected filter is $\bot^{\mathscr{F}}$.

$\square$

PROPOSITION 2406. Connected filters regarding

1°. $1^{\mathsf{FCD}(A)}$;
2°. $1^{\mathsf{RLD}(A)}$

are exactly ultrafilters and the improper filter.

PROOF. 1. That ultrafilters are connected follows from the fact that for every non-least $\mathcal{X}$, $\mathcal{Y}$ such that $\mathcal{X} \sqcup \mathcal{Y} = \mathcal{A}$ we have $\mathcal{X} = \mathcal{Y} = \mathcal{A}$ and thus $\mathcal{X} \; [1^{\mathsf{FCD}(A)}] \; \mathcal{Y}$. So ultrafilters are connected; so is improper filter too, because improper filter is always connected.

It remains to prove that filters containing more than one distinct ultrafilter are not connected. Really let distinct ultrafilters $a, b \in \operatorname{atoms}\mathcal{A}$. Then not $a \; [1^{\mathsf{FCD}(A)}] \; b$. Thus $\mathcal{A}$ is not connected.

2. A filter a is connected iff $S_1^*(1^{\mathsf{RLD}(A)} \sqcap (a \times^{\mathsf{RLD}} a)) \sqsupseteq a \times^{\mathsf{RLD}} a$ that is iff $S_1^*(\mathrm{id}_a^{\mathsf{RLD}}) \sqsupseteq a \times^{\mathsf{RLD}} a$,

$\bigsqcap_{F \in \operatorname{up}\mathrm{id}_a^{\mathsf{RLD}}} S_1(F) \sqsupseteq a \times^{\mathsf{RLD}} a$ what by properties of generalized filter bases is equivalent to $\bigsqcap_{A \in \operatorname{up} a} S_1(\mathrm{id}_A) \sqsupseteq a \times^{\mathsf{RLD}} a$; $\bigsqcap_{A \in \operatorname{up} a} \mathrm{id}_A \sqsupseteq a \times^{\mathsf{RLD}} a$; $\mathrm{id}_a^{\mathsf{RLD}} \sqsupseteq a \times^{\mathsf{RLD}} a$. This is true exactly for ultrafilters and the improper filter. $\square$

DEFINITION 2407. A *path* regarding funcoid μ is a tuple $p_0, \dots, p_n$ ($n \in \mathbb{N}$) of atomic filters such that $p_i \; [\mu] \; p_{i+1}$ for every $i = 0, \dots, n-1$.

The number n is called *path length*.

A path is *between* atomic filters a and b iff $p_0 = a$ and $p_n = b$.

EXAMPLE 2408. $\mu \sqsupseteq \mathrm{id}_{\mathcal{A}}^{\mathsf{FCD}}$ is not necessary for a filter $\mathcal{A}$ to be connected regarding a funcoid μ. Moreover $\mu \sqsupseteq 1^{\mathsf{FCD}}$ is not necessary for a filter $\top$ to be connected regarding a funcoid μ.

Proof. For counterexample take $\mu = \top \setminus 1$.

$\langle \mu \rangle \{x\} = \top \setminus \{x\}$ (thus $\mu \not\sqsupseteq 1^{\mathsf{FCD}}$) and $\langle \mu \rangle a = \top$ for a nontrivial ultrafilter a.

Let $\mathcal{X}, \mathcal{Y} \in \mathscr{F}(\operatorname{Ob} \mu) \setminus \{\bot\}$ and $\mathcal{X} \sqcup \mathcal{Y} = \top$. If $\mathcal{X}$ is a trivial ultrafilter then $\langle \mu \rangle \mathcal{X} = \top \setminus \{x\}$ adn thus $\langle \mu \rangle \mathcal{X} \not\asymp \mathcal{Y}$, otherwise $\langle \mu \rangle \mathcal{X} \not\asymp \mathcal{Y}$. So in any case $\mathcal{X} \mathrel{[\mu]} \mathcal{Y}$. Funcoid μ is connected. □

Proposition 2409. If there is a nonzero-length path regarding μ in the filter $\mathcal{A}$ between any two its atomic subfilters, then it is connected regarding μ.

Proof. Let $\mathcal{X} \sqcup \mathcal{Y} = \mathcal{A}$, $\mathcal{X} \neq \bot$, $\mathcal{Y} \neq \bot$. Let $p_0, \ldots, p_n$ ($n \geq 1$) be a path in $\mathcal{A}$ and $p_0 \in \operatorname{atoms} \mathcal{X}$ and $p_n \in \operatorname{atoms} \mathcal{Y}$. Then (take $k = \min\{ i \in \{0, \ldots, n-1\} \mid p_{i+1} \in \operatorname{atoms} \mathcal{Y}\}$) there are p_k, p_{k+1} such that $p_k \in \operatorname{atoms} \mathcal{X}$, $p_{k+1} \in \operatorname{atoms} \mathcal{Y}$. But $p_k \mathrel{[\mu]} p_{k+1}$ by definition of path. Thus $\mathcal{X} \mathrel{[\mu]} \mathcal{Y}$. □

Proposition 2410. If a filter $\mathcal{A}$ is connected regarding funcoid μ reflexive on $\mathcal{A}$ then it is connected regarding every μ^n for $n \in \mathbb{Z}_+$.

Proof. Let $\mathcal{X} \sqcup \mathcal{Y} = \mathcal{A}$, $\mathcal{X} \neq \bot$, $\mathcal{Y} \neq \bot$. We have $\langle \mu \rangle \mathcal{X} \not\asymp \mathcal{Y}$.

Then $\langle \mu \rangle \mathcal{X} \not\sqsubseteq \mathcal{X}$; therefore by reflexivity $\langle \mu \rangle \mathcal{X} \sqsupset \mathcal{X}$. Repeating this step we get $\langle \mu \rangle \langle \mu \rangle \mathcal{X} \sqsupset \mathcal{X}$ that is $\langle \mu^2 \rangle \mathcal{X} \sqsupset \mathcal{X}$, etc.

We have $\langle \mu^n \rangle \mathcal{X} \sqsupset \mathcal{X}$ and thus $\langle \mu^n \rangle \mathcal{X} \not\asymp \mathcal{Y}$ that is $\mathcal{X} \mathrel{[\mu^n]} \mathcal{Y}$. □

Example 2411. Connected funcoid without a path between given ultrafilters.

Proof. Consider $|\mathbb{R}|$. It is connected (prove!) but there is no path (prove!) between two distinct singletons. □

Theorem 2412. If meet of two connected (regarding a funcoid) filters is nonleast, then their join is connected.

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be intersecting filters, both connected regarding an endofuncoid μ. Let $\mathcal{X} \sqcup \mathcal{Y} = \mathcal{A} \sqcup \mathcal{B}$ for proper filters $\mathcal{X}, \mathcal{Y}$. Then either $\mathcal{X}$ or $\mathcal{Y}$ intersects $\mathcal{A} \sqcap \mathcal{B}$. Without loss of generality assume $\mathcal{X} \sqcap \mathcal{A} \sqcap \mathcal{B} \neq \bot$. Also $\mathcal{Y}$ intersects either $\mathcal{A}$ or $\mathcal{B}$. Without loss of generality assume $\mathcal{Y} \sqcap \mathcal{A} \neq \bot$.

Note $\mathcal{X} \sqcap \mathcal{A} \neq \bot$.

We have $(\mathcal{X} \sqcap \mathcal{A}) \sqcup (\mathcal{Y} \sqcap \mathcal{A}) = (\mathcal{X} \sqcup \mathcal{Y}) \sqcap \mathcal{A} = (\mathcal{A} \sqcup \mathcal{B}) \sqcap \mathcal{A} = \mathcal{A}$. So $\mathcal{X} \sqcap \mathcal{A} \mathrel{[\mu]} \mathcal{Y} \sqcap \mathcal{A}$ because $\mathcal{A}$ is connected, consequently $\mathcal{X} \mathrel{[\mu]} \mathcal{Y}$ that is $\mathcal{A} \sqcup \mathcal{B}$ is connected. □

Theorem 2413. If meet of two connected (regarding a reloid) filters is nonempty, then their join is connected.

Proof. Let $S_1^*(\mu \sqcap (\mathcal{A} \times \mathcal{A})) = \mathcal{A} \times \mathcal{A}$; $S_1^*(\mu \sqcap (\mathcal{B} \times \mathcal{B})) = \mathcal{B} \times \mathcal{B}$ for filters $\mathcal{A} \not\asymp \mathcal{B}$.

$S_1^*(\mu \sqcap ((\mathcal{A} \sqcup \mathcal{B}) \times (\mathcal{A} \sqcup \mathcal{B}))) = S_1^*(\mu \sqcap ((\mathcal{A} \times \mathcal{A}) \sqcup (\mathcal{B} \times \mathcal{B}) \sqcup (\mathcal{A} \times \mathcal{B}) \sqcup (\mathcal{B} \times \mathcal{A}))) \sqsupseteq S_1^*(\mu \sqcap (\mathcal{A} \times \mathcal{A})) \sqcup S_1^*(\mu \sqcap (\mathcal{B} \times \mathcal{B})) \sqsupseteq (\mathcal{A} \times \mathcal{A}) \sqcup (\mathcal{B} \times \mathcal{B})$.

Let for example $x \in \operatorname{atoms} \mathcal{A}$. Then $\langle S_1^*(\mu \sqcap ((\mathcal{A} \sqcup \mathcal{B}) \times (\mathcal{A} \sqcup \mathcal{B}))) \rangle x \sqsupseteq \mathcal{A}$ and (taking into account $\mathcal{A} \not\asymp \mathcal{B}$):

$$\langle \mu \sqcap ((\mathcal{A} \sqcup \mathcal{B}) \times (\mathcal{A} \sqcup \mathcal{B})) \rangle \langle S_1^*(\mu \sqcap ((\mathcal{A} \sqcup \mathcal{B}) \times (\mathcal{A} \sqcup \mathcal{B}))) \rangle x \sqsupseteq \mathcal{B}.$$

Thus $\langle S_1^*(\mu \sqcap ((\mathcal{A} \sqcup \mathcal{B}) \times (\mathcal{A} \sqcup \mathcal{B}))) \rangle x \sqsupseteq \mathcal{A}$ and $\langle S_1^*(\mu \sqcap ((\mathcal{A} \sqcup \mathcal{B}) \times (\mathcal{A} \sqcup \mathcal{B}))) \rangle x \sqsupseteq \mathcal{B}$ for every ultrafilter $x \in \operatorname{atoms}(\mathcal{A} \sqcup \mathcal{B})$, that is $\langle S_1^*(\mu \sqcap ((\mathcal{A} \sqcup \mathcal{B}) \times (\mathcal{A} \sqcup \mathcal{B}))) \rangle x \sqsupseteq \mathcal{A} \sqcup \mathcal{B}$. So $S_1^*(\mu \sqcap ((\mathcal{A} \sqcup \mathcal{B}) \times (\mathcal{A} \sqcup \mathcal{B}))) \sqsupseteq \mathcal{A} \sqcup \mathcal{B}$ that is $\mathcal{A} \sqcup \mathcal{B}$ is connected. □

Corollary 2414. Distinct connected components (for both a funcoid or a reloid) don't intersect.

Proof. If connected components $\mathcal{A} \neq \mathcal{B}$ intersect, then $\mathcal{A} \sqcup \mathcal{B}$ is a connected filter and either $\mathcal{A} \sqcup \mathcal{B} \sqsupset \mathcal{A}$ or $\mathcal{A} \sqcup \mathcal{B} \sqsupset \mathcal{B}$ what contradicts to the definition of connected components. □

If we add the requirement $\mathcal{X} \asymp \mathcal{Y}$ to the definition of connected regarding funcoid, it is nonequivalent. Proof??: Consider connectedness of an ultrafilter.

Proposition 2415. $S(\mu) = S_1(\mu \sqcup 1)$ if μ is an endorelation, endofuncoid, or endoreloid. FiXme: for S^*, too.

Proof. By proved above $(\mu \sqcup 1)^n = 1 \sqcup \mu \sqcup \ldots \sqcup \mu^n$.
Thus $S_1(\mu \sqcup 1) = (1 \sqcup \mu) \sqcup (1 \sqcup \mu \sqcup \mu^2) \sqcup \ldots = 1 \sqcup \mu \sqcup \mu^2 \sqcup \ldots = S(\mu)$. $\square$

FiXme: also algebraic properties of S_1 and S_1^*

Theorem 2416. FiXme: Move this theorem in the book, $\mathcal{X} \left[\bigsqcap S\right] \mathcal{Y} \Leftrightarrow \forall f \in S : \mathcal{X} [f] \mathcal{Y}$ if S is a generalized filter base.

Proof. $\mathcal{X} \left[\bigsqcap S\right] \mathcal{Y} \Leftrightarrow (\mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y}) \sqcap \bigsqcap S \neq \bot \Leftrightarrow \bigsqcap_{f \in S} f \sqcap (\mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y}) \neq \bot \Leftrightarrow$ (by properties of generalized filter bases) $\Leftrightarrow \forall f \in S : f \sqcap (\mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y}) \neq \bot \Leftrightarrow \forall f \in S : \mathcal{X} [f] \mathcal{Y}$. $\square$

Theorem 2417. The following are pairwise equivalent for a funcoid μ and filter $\mathcal{A}$:

1°. $\mathcal{A}$ is connected regarding funcoid μ
2°. $\mathcal{A}$ is connected regarding every funcoid in $\operatorname{up} \mu$.
3°. $\mathcal{A}$ is connected regarding every funcoid in $\operatorname{up}^\Gamma \mu$.

Proof. TODO: "Connectedness" should be moved after "Funcoids are filters" to use Γ in this proof.

1⇒2⇒3. Obvious.

3⇒1. Let $\mathcal{X}, \mathcal{Y} \in \mathscr{F}(\operatorname{Ob} \mu)$ and $\mathcal{X} \sqcup \mathcal{Y} = \mathcal{A}$. Then $\forall f \in \operatorname{up}^\Gamma \mu : \mathcal{X} [f] \mathcal{Y}$. Therefore by the theorem ?? $\mathcal{X} \left[\bigsqcap \operatorname{up}^\Gamma \mu\right] \mathcal{Y}$ that is $\mathcal{X} [\mu] \mathcal{Y}$. So $\mathcal{A}$ is connected regarding μ. $\square$

Conjecture 2418. For a **Rel**-morphism F and a filter $\mathcal{A}$ the following are pairwise equivalent:

1°. $\mathcal{A}$ is connected regarding $\uparrow^{\mathsf{FCD}} F$.
2°. $\mathcal{A}$ is connected regarding $\uparrow^{\mathsf{RLD}} F$.
3°. there is a F-path in $\mathcal{A}$ for every two ultrafilters $a, b \in \operatorname{atoms} \mathcal{A}$.

Proposed counterexample against $\mathcal{A}$ is connected regarding f iff it is connected regarding $(\mathsf{FCD})f$: $f = \mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{A}$. First calculate $(\mathcal{B} \times^{\mathsf{RLD}}_F \mathcal{C}) \circ (\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B})$ (and also for oblique product).

Trying to calculate $(\mathcal{B} \times^{\mathsf{RLD}}_F \mathcal{C}) \circ (\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B})$:

Lemma 2419. There are such filters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and binary relation h that

$$h \sqsupseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{C} \wedge \neg \exists g \in \mathbf{Rel} : (g \sqsupseteq \mathcal{B} \times^{\mathsf{FCD}} \mathcal{C} \wedge h \sqsupseteq g \circ (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})).$$

Proof. Take $\mathcal{A}$ a principal filter, $\mathcal{B}$ a trivial ultrafilter and $h \sqsupseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{C}$ such that $h \notin \operatorname{up}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{C})$. (It exists because $\mathcal{A} \times^{\mathsf{RLD}} \mathcal{C} \neq \mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{C}$.

Suppose that $g \sqsupseteq \mathcal{B} \times^{\mathsf{FCD}} \mathcal{C}$. Then there is $C \in \operatorname{up} \mathcal{C}$ such that $g \sqsupseteq \mathcal{B} \times C$. Therefore $g \circ (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = \mathcal{A} \times^{\mathsf{FCD}} \langle g \rangle \mathcal{B} \sqsupseteq \mathcal{A} \times^{\mathsf{FCD}} C = \mathcal{A} \times C$.)

But $h \notin \operatorname{up}(\mathcal{A} \times^{\mathsf{RLD}} C) = \operatorname{up}(\mathcal{A} \times C)$. Thus $h \not\sqsupseteq g \circ (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})$. $\square$

Corollary 2420. There are such filters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and binary relation h that

$$h \sqsupseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{C} \wedge \neg \exists f, g \in \mathbf{Rel} : (f \sqsupseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \wedge g \sqsupseteq \mathcal{B} \times^{\mathsf{FCD}} \mathcal{C} \wedge h \sqsupseteq g \circ f).$$

Proposition 2421. $(\mathcal{B} \times^{\mathsf{RLD}}_F \mathcal{C}) \circ (\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B}) \neq \mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{C}$ for some proper filters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$.

PROOF. FiXme: The proof is erroneous.

Take (lemma) $h \in \mathrm{up}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{C})$ such that for every $f \in \mathrm{up}(\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{C})$, $g \in \mathrm{up}(\mathcal{B} \times^{\mathsf{RLD}}_F \mathcal{C})$ we have $h \not\supseteq g \circ f$.

We have $h \in \mathrm{up}(\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{C})$ and for every $f \in \mathrm{up}(\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{C})$, $g \in \mathrm{up}(\mathcal{B} \times^{\mathsf{RLD}}_F \mathcal{C})$ we have [error] $h \not\supseteq g \circ f$.

Thus $\mathrm{up}((\mathcal{B} \times^{\mathsf{RLD}}_F \mathcal{C}) \circ (\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B})) \neq \mathrm{up}(\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{C})$. $\square$

CHAPTER 15

Relationships are pointfree funcoids

THEOREM 2422. $((\mathsf{FCD}), (\mathsf{RLD})_{\mathrm{in}})$ are components of a complete pointfree funcoid.

PROOF. For every ultrafilters x and y we have $x\ [(\mathsf{FCD})(f \sqcap (\mathsf{RLD})_{\mathrm{in}} g)]\ y \Leftrightarrow x \times^{\mathsf{RLD}} y \not\asymp f \sqcap (\mathsf{RLD})_{\mathrm{in}} g \Leftrightarrow x \times^{\mathsf{RLD}} y \sqsubseteq (\mathsf{RLD})_{\mathrm{in}} g \wedge x \times^{\mathsf{RLD}} y \not\asymp f \sqcap (\mathsf{RLD})_{\mathrm{in}} g \Leftrightarrow x \times^{\mathsf{FCD}} y \in \operatorname{atoms} g : x \times^{\mathsf{RLD}} y \not\asymp f \sqcap (\mathsf{RLD})_{\mathrm{in}} g \Leftrightarrow x \times^{\mathsf{FCD}} y \in \operatorname{atoms} g : x \times^{\mathsf{RLD}} y \not\asymp f \Leftrightarrow x \times^{\mathsf{FCD}} y \in \operatorname{atoms} g \wedge x \times^{\mathsf{FCD}} y \sqsubseteq (\mathsf{FCD}) f \Leftrightarrow x\ [g \sqcap (\mathsf{FCD}) f]\ y$.

Thus $(\mathsf{FCD})(f \sqcap (\mathsf{RLD})_{\mathrm{in}} g) = g \sqcap (\mathsf{FCD}) f$. Consequently $f \sqcap (\mathsf{RLD})_{\mathrm{in}} g = \bot \Leftrightarrow g \sqcap (\mathsf{FCD}) f = \bot$ that is $g \not\asymp (\mathsf{FCD}) f \Leftrightarrow f \not\asymp (\mathsf{RLD})_{\mathrm{in}} g$.

It is complete by theorem 1198. □

We will also prove in another way that (FCD), $(\mathsf{RLD})_{\mathrm{in}}$ are components of pointfree funcoids:

THEOREM 2423. $(\mathsf{RLD})_{\mathrm{in}}$ is a component of a pointfree funcoid (between filters on boolean lattices).

PROOF. Consider the pointfree funcoid $\mathscr{R}$ defined by the formula $\langle \mathscr{R} \rangle^* F = (\mathsf{RLD})_{\mathrm{in}} F$ for binary relations F (obviously it does exists). Then $\langle \mathscr{R} \rangle f = \langle \mathscr{R} \rangle \bigsqcap^{\mathsf{FCD}} \operatorname{up}^{\Gamma} f = \bigsqcap^{\mathsf{RLD}}_{F \in \operatorname{up}^{\Gamma} f} \langle \mathscr{R} \rangle^* F = \bigsqcap^{\mathsf{RLD}}_{F \in \operatorname{up}^{\Gamma} f} (\mathsf{RLD})_{\mathrm{in}} F = (\mathsf{RLD})_{\mathrm{in}} \bigsqcap^{\mathsf{FCD}}_{F \in \operatorname{up}^{\Gamma} f} F = (\mathsf{RLD})_{\mathrm{in}} f$. □

THEOREM 2424. (FCD) is a component of a complete pointfree funcoid (between filters on boolean lattices).

PROOF. Consider the pointfree funcoid $\mathscr{Q}$ defined by the formula $\langle \mathscr{Q} \rangle^* F = (\mathsf{FCD}) F$ for binary relations F (obviously it does exists). Then $\langle \mathscr{Q} \rangle f = \langle \mathscr{Q} \rangle \bigsqcap^{\mathsf{RLD}} \operatorname{up} f =$ (because $\operatorname{up} f$ is a filter base) $= \bigsqcap^{\mathsf{FCD}}_{F \in \operatorname{up} f} \langle \mathscr{Q} \rangle^* F = \bigsqcap^{\mathsf{FCD}}_{F \in \operatorname{up} f} (\mathsf{FCD}) F = \bigsqcap^{\mathsf{FCD}}_{F \in \operatorname{up} f} F = \bigsqcap^{\mathsf{FCD}} \operatorname{up} f = (\mathsf{FCD}) f$. □

PROPOSITION 2425. $(\mathsf{FCD}) \bigsqcap S = \bigsqcap_{f \in S} (\mathsf{FCD}) f$ if S is a filter base of reloids (with the same sources and destinations).

PROOF. Theorem 831. □

CONJECTURE 2426. $(\mathsf{RLD})_{\mathrm{in}} \bigsqcap S = \bigsqcap_{f \in S} (\mathsf{RLD})_{\mathrm{in}} f$ if S is a filter base of funcoids (with the same sources and destinations).

CHAPTER 16

Manifolds and surfaces

1. Sides of a surface

Definition 2427. Let μ be an endofuncoid on a set U. *Surface side* of a set $T \subseteq \operatorname{Ob}\mu$ is a connected component (regarding μ) of the filter $(\langle\mu\rangle^* T) \setminus T$. FiXme: μ is used twice in this definition. We may generalize for two different funcoids instead.

Keep in mind that the above definition may work nicely if μ is a complete funcoid induced by a topological space.

Example 2428. For an $\mathbb{R}^{n-1}$ subspace T of a $\mathbb{R}^n$ $(n \geq 1)$ euclidean space and the complete funcoid μ induced by the usual topology:

1°. T has exactly two surface sides.

2°. The filter $\langle\mu\rangle^* @\{a\} \setminus T$ (for every $a \in T$) has exactly two connected components.

Proof. Without loss of generality assume that

$$T = \left\{ \frac{(x_0, x_1, \ldots, x_{n-2}, 0)}{x_0, x_1, \ldots, x_{n-2} \in \mathbb{R}} \right\}; \quad a = (0, \ldots, 0).$$

We have

$$\langle\mu\rangle^* @\{a\} = \left(\uparrow \left\{ \frac{v \in \mathbb{R}^n}{v_{n-1} > 0} \right\} \sqcap \langle\mu\rangle^* @\{a\} \right) \sqcup \left(\uparrow \left\{ \frac{v \in \mathbb{R}^n}{v_{n-1} < 0} \right\} \sqcap \langle\mu\rangle^* @\{a\} \right).$$

Let us prove that $\uparrow \left\{ \frac{v \in \mathbb{R}^n}{v_{n-1} > 0} \right\} \sqcap \langle\mu\rangle^* @\{a\}$ and $\uparrow \left\{ \frac{v \in \mathbb{R}^n}{v_{n-1} < 0} \right\} \sqcap \langle\mu\rangle^* @\{a\}$ are connected components.

?? □

1.1. Special points.

We will start from the example of open $T = \left\{ \frac{(x,y,0)}{x^2+y^2<1} \right\}$ and closed $T = \left\{ \frac{(x,y,0)}{x^2+y^2 \leq 1} \right\}$ disks in $\mathbb{R}^3$.

Exercise 2429. Prove that open disk (in a usual 3-dimensional space) has two surface sides and closed disk has one surface side.

2. Special points

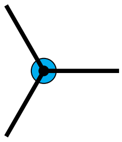
Definition 2430. *Surface cardinality* of a point a (an element of the set $\operatorname{Ob}\mu$) is the cardinality of the set of connected components of the filter $\langle\mu\rangle^*\{a\} \setminus T$.

Definition 2431. *Cardinality regular point* is a point a, which has a neighborhood $(X \in \operatorname{up}\langle\mu\rangle^*\{a\})$ such that all points $x \in X \cap T$ are of the same surface cardinality as the point a.

Cardinality special point is a point which is not cardinality regular.

Definition 2432. *Isomorphism regular point* is a point a, which has a neighborhood $(X \in \operatorname{up}\langle\mu\rangle^*\{a\})$ such that for all points $x \in X \cap T$ the filter $\langle\mu\rangle^*\{a\}$ is isomorphic to $\langle\mu\rangle^*\{x\}$.

Isomorphism special point is a point which is not isomorphism regular.



(A) Three surface sides

(B) Two surface sides

FIGURE 1. Examples of surface cardinality

FiXme: Try to replace isomorphism f with some kind of filter embedding.

Consider the dihedral angle T produced by two half-planes. Are the points of intersection of the half-planes isomorphism-special? (They should not be considered special. If they are special, this is a probably flaw in the definition of isomorphism special.)

Consider union T of two intersecting lines on a plane. The intersection may be considered as a special point, because it has more connected components that the rest. We don't want to consider it special, however. We can restrict to consider special only points which have less connected components (rather than more) to correct this trouble. Also try to define it with some kind of morphisms of filters instead of isomorphism as in isomorphism-special.

EXERCISE 2433. Excluding special points (either cardinality or isomorphism) from closed disk produces open disk.

Let us note that special points of closed disk have surface cardinality 1 which is less than surface cardinality (2) of regular points. So, it is a conceivable idea to consider special points which have lesser surface cardinality than nearby points.

Consider the following two subsets of a plane (the lines are the set T, the small black blob is the point a, and the cyan blob symbolizes the filter $(\langle\mu\rangle^*\{a\}) \setminus T$):

For one of the sets surface cardinality of a is 3 and for another it is 2.

Now define *shift special points*.

Let I be an interval on $\mathbb{R}$ (containing zero?)

A point a is *shift special* if there exists a transformation (that is a continuous function $f : I \times \mu \to \mu$ such that:

1°. $f(0)$ is identity. FiXme: Is this condition needed?
2°. for every sufficiently small $\epsilon > 0$ we have $f(\epsilon, a) \in T$;
3°. there is $\epsilon > 0$ such that for every $0 < \epsilon' < \epsilon$ we have $f(\epsilon')$ being not continuous at a regarding complete funcoid defined by the function $x \mapsto \langle\mu\rangle^*\{x\} \setminus T$.

We may consider to additonally require that every $f(\epsilon)$ is isomorphism of funcoids.

EXAMPLE 2434. T is disk $\left\{\frac{(x,y,0)}{x^2+y^2\leq 1}\right\}$. f is the contraction $(\epsilon, v) \mapsto \frac{1}{1+\epsilon}v$. $a = (1, 0, 0)$.

In the usual topology f is continuous. In $x \mapsto \langle\mu\rangle^*\{x\} \setminus T$ we have the function $\epsilon \mapsto f(\epsilon)$ not continuous at zero. So a is a shift special point.

PROOF. $f(0)(v) = v$. Thus $\langle f(0)\rangle(\langle\mu\rangle^*\{a\} \setminus T) = \langle\mu\rangle^*\{a\} \setminus T$ intersects the plane $Z = 0$. But $f(0, a)$

?? □

QUESTION 2435. Can we exclude real numbers from the play?

QUESTION 2436. How cardinality special points, isomorphism special points and shift special points are related with each others?

QUESTION 2437. How the number of surface sides is related with usual surface sides for manifolds? https://en.wikipedia.org/wiki/Orientability#Orientability_of_manifolds

REMARK 2438. Manifolds have no special points. (Prove!)

Prove that 2-manifold image which special points removed has the same number of sides as the defined above.

Another way to define special points: A special point is a point such that $T \sqcap \langle\mu\rangle\{a\}$ is not isomorphic to $T \sqcap \langle\mu\rangle\{x\}$ for nearby points x. Consider replacement of isomorphism with injection, surjection, etc. here and above.

How many sides has in $\mathbb{R}^3$ a plane without one point?

Easy way to spot special points: They are boundary points in the topology (or funcoid) induced on T. Alternatively we can consider points whose neighborhood in T is different (as non-isomorphic or maybe non-injective or non-surjective or like this) than of nearby points. Thus another way to remove special points: use interior funcoid.

https://math.stackexchange.com/q/2836833/4876

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